Towards Optimal Kron-based Reduction Of Networks (Opti-KRON) for the Electric Power Grid

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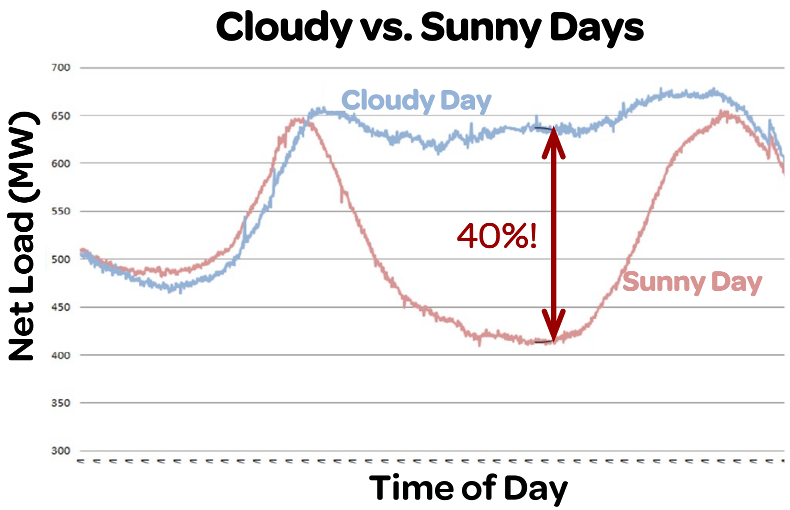
Control over networks is challenging

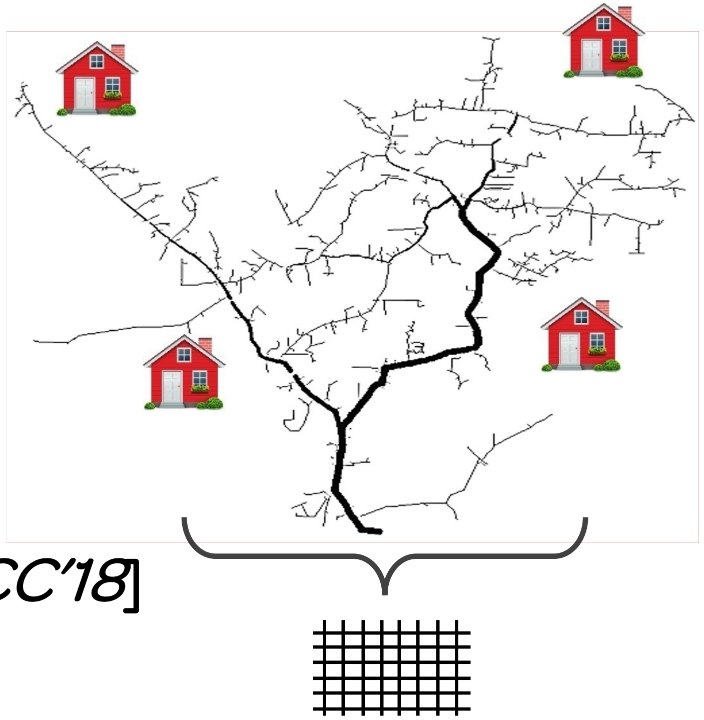
- **Optimal control of electrical networks is important...**
  - distributed energy resource (DER) control
  - electric vehicle charging infrastructure & control,
  - distribution grid asset and topology control
  - regulation of frequency/voltage (avoid over-voltages)
  - dynamic/time of use incentive signals

...but computationally HARD!!!
  - networks have 10,000s of vertices and edges
  - multiple unbalanced phases
  - nonlinear network physics: AC power flow
  - nonconvex solution spaces...



How have others solved HARD problem? (lit review)

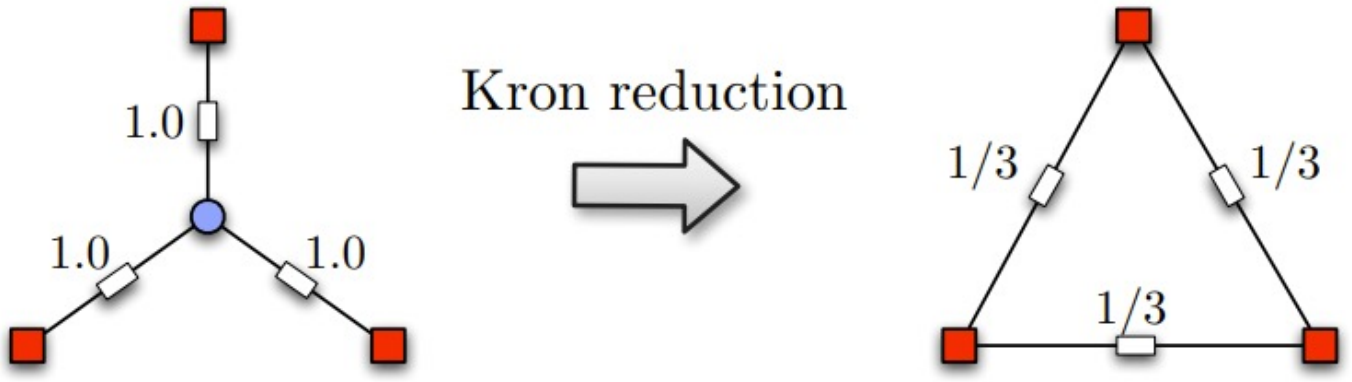
- **Network linearizations, convexifications, decompositions**
  - “DC” power flow approximation [*Stott et al, TPWRS’07*]
  - Convex *relaxation* of AC power flow [*Low, TCNS’14*]
  - Convex *restriction* of AC power flow [*Lee et al, TPWRS’21*]
  - Distributed optimization [*Dall’Anese et al, TSG’18*]



- **Network *compressions***
  - Zonal clustering + admittance tuning [*Fortenbacher et al, PSCC’18*]
  - Bus clustering + quadratic optimization [*Shi et al, TPWRS’15*]
  - PTDF-based clustering [*Oh, TPWRS’10*]
  - Kron reduction (Ward equivalency) [*Dorfler et al, TCS’13*]
  - Partitioning and Kron-reduction for planning [*Ploussard, PhD’19*]
  - Kron reduction for 3-ph grid OPF acceleration [*Almassalkhi et al, Energies’20*]

What is a Kron Reduction?

- "Reduction of an electrical network via a Schur complement"



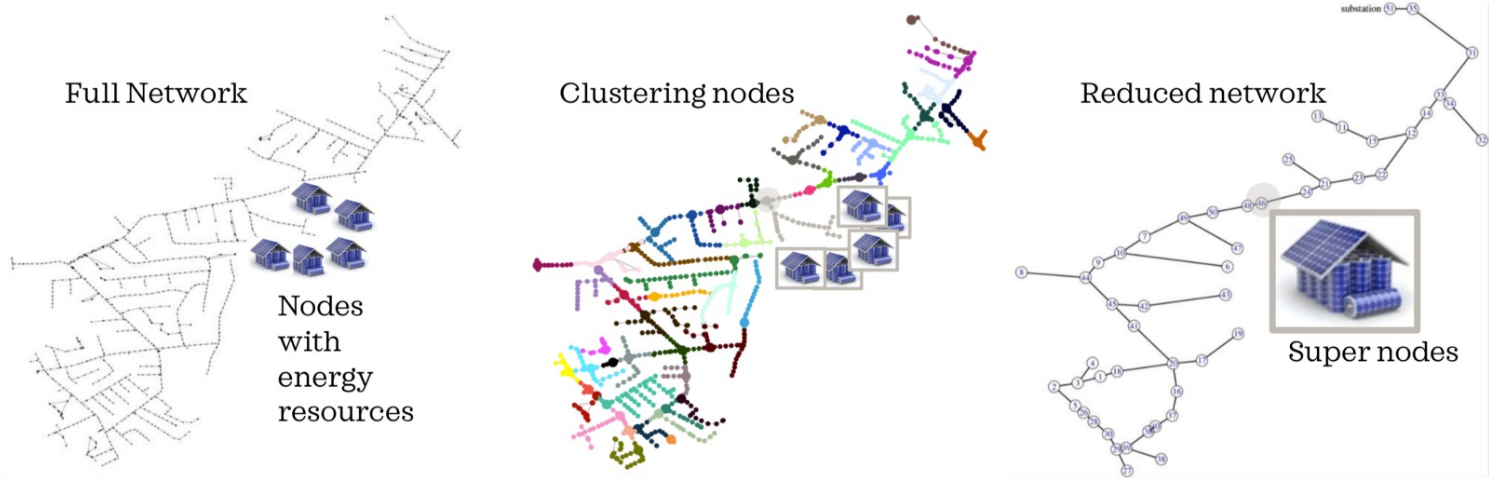
[Dorfler et al, TCS'13]

- How? Assume the blue node has 0 current injection:

$$\left[\begin{array}{c} I_k \\ \hline 0 \end{array}\right] = \left[\begin{array}{c|c} Y_{b1} & Y_{b2} \\ \hline Y_{b3} & Y_{b4} \end{array}\right] \left[\begin{array}{c} V_k \\ \hline V_r \end{array}\right] \quad \Rightarrow I_k = \underbrace{(Y_{b1} - Y_{b2} Y_{b4}^{-1} Y_{b3})}_{Y_{\text{kron}}} V_k$$

Opti-KRON: Optimal Kron-based Reduction Of Networks

- **Many network compression algorithms are built around clustering, nodal aggregation, and Kron reduction, e.g.,**



[Almassalkhi et al., 2020]

  – But how should buses be clustered…? In the literature, heuristics are used.

- This paper asks a simple, but interesting question:

Is there an optimal Kron reduction?

Opti-KRON: Optimal Kron-based Reduction Of Networks

Formulating Opti-KRON with binary selection variables:

Consider a network with *m* edges (i.e., lines/brances) and *n* vertices (i.e., nodes/buses)

$$I = Y_b V \qquad Y_b = E^T Y_l E + Y_s$$ (E is mxn adj. matrix)

Consider the Kron impedance matrix (instead of admittance):

$$Z_b I = V \iff \left[\begin{array}{c|c} Z_{b1} & Z_{b2} \\ \hline Z_{b3} & Z_{b4} \end{array}\right] \left[\begin{array}{c} I_k \\ \hline 0 \end{array}\right] = \left[\begin{array}{c} V_k \\ \hline V_r \end{array}\right] \Rightarrow Z_{b1} = Y_{\text{kron}}^{-1} =: Z_{\text{kron}}$$

Let nodal selection $s \in \{0,1\}^n$ variable be defined as s_i =0, if node i reduced; else =1 (kept)

Then let $S = \text{diag}\{s\}$ and the "generalized" (full-size) Kron impedance matrix is given by:

$$Z_{\hat{K}} = S Z_b S$$ e.g., $Z_{\hat{K}} = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} Z_{b1} & Z_{b2} \\ \hline Z_{b3} & Z_{b4} \end{array}\right] \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} Z_{b1} & 0 \\ \hline 0 & 0 \end{array}\right]$

Opti-KRON: Optimal Kron-based Reduction Of Networks

Formulating Opti-KRON with binary selection variables:

Need to map reduced nodal injections to kept nodes: Aggregation matrix, $A \in \{0,1\}^{n \times n}$

$$A_{ij} = \begin{cases} 1 & \text{, if nodal injection current from bus } j \text{ is placed at bus } i \\ 0 & \text{, otherwise} \end{cases}$$

Now, we can embed Kron reduction rules with Selector and Aggregator matrices

- Any assigned current can only be assigned once: $\sum_i A_{ij} = 1$
- No current can be assigned at reduced nodes: $\sum_j A_{ij} \leq M_{\text{agg}} S_{ii}$ (Big-M formulation)
- Kept (non-reduced) nodes cannot assign their current: $A_{ij} = S_{ii}$

Opti-KRON: Optimal Kron-based Reduction Of Networks

Formulating Opti-KRON with binary selection variables:

Full (naïve) Kron formulation is given by

$$Z_{\text{kron}} = S Z_b S$$
$$I_{\text{kron}} = A I$$
$$V_{\text{kron}} = Z_{\text{kron}} I_{\text{kron}}$$

$$\Longleftrightarrow$$

$$V_K = S Z_b S A I$$
$$s_i,\ A_{i,j} \in \{0,1\}$$
$$S = \text{diag}(s)$$
$$\sum_i A_{i,j} = 1$$
$$\sum_j A_{i,j} \leq M_b S_{i,i}$$
$$S_{i,i} = A_{i,i}.$$

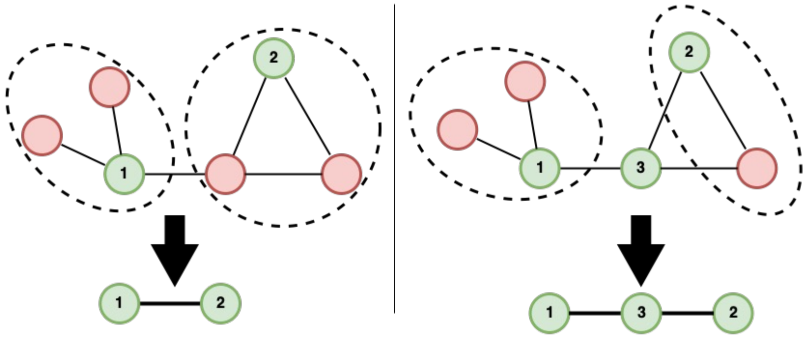
Opti-KRON: Optimal Kron-based Reduction Of Networks

Our approach:

1. Collect a library of representation network power flow solutions (V,I)
2. Define an objective function which
   I. rewards network compression,
   II. and penalizes network error

$$\mathscr{L} = \underbrace{\left\| V_{K,i} - V_{j \in C_{K,i}} \right\|_{\infty}}_{\text{error}} - \alpha \underbrace{\frac{1}{n} \sum_{j=1}^{n} (1 - s_j)}_{\text{complexity}}$$

3. Pose and solve a mixed integer linear (MILP) program which chooses how to aggregate electrical loads onto "super nodes" for an optimal Kron reduction



Opti-KRON: Optimal Kron-based Reduction Of Networks

Binary-integer Opti-KRON formulation

$$\min_{s,A} \quad \left\| V_{K,i} - V_{j \in C_{K,i}} \right\|_\infty - \frac{\alpha}{n} \sum_{j=1}^{n} (1 - s_j)$$

Objective: error vs. complexity

$$\text{s.t.} \quad V_K = S Z_b S A I$$

CHALLENGE!! Cubic binary terms (2 x S and 1 x A)

$$s_i, \; A_{i,j} \in \{0, 1\}$$

$$S = \mathrm{diag}(s)$$

$$\sum_i A_{i,j} = 1$$

$$\sum_j A_{i,j} \leq M_b S_{i,i}$$

$$S_{i,i} = A_{i,i}.$$

- **Step 1**: $SA = A$ (proof in paper)
- **Step 2**: $S Z_b A I \Rightarrow Z_b A I + \text{Big-M}$ (proof in paper)
- **Step 3**: Formulation permits non-physical current assignments!
  - Limit aggregation to only neighbors: use Graph Laplacian
  - Add constraint:

$$A_{ij} \leq |E^T E|_{ij}$$

See paper for details and computational tricks!

Opti-KRON: Optimal Kron-based Reduction Of Networks

New tractable mixed-integer Opti-KRON formulation

$$\min_{s,A,\delta} \ \delta - \frac{\alpha}{n}\sum_{j=1}^{n}(1-s_j)$$

Objective: error vs. complexity

$$\begin{aligned}
\text{s.t.} \ & \tilde{V}_{K,i} - V_{j\in C_{K,i}} \leq \delta + M_v(1-A_{i,j}), \quad \forall i,j \\
& V_{j\in C_{K,i}} - \tilde{V}_{K,i} \leq \delta + M_v(1-A_{i,j}), \quad \forall i,j \\
& \tilde{V}_K = Z_b A I \\
& \sum_i A_{i,j} = 1 \quad \forall j \\
& \sum_j A_{i,j} \leq M_b S_{i,i} \quad \forall i \\
& A_{i,j} \leq |E^T E|_{i,j} \quad \forall i,j \\
& S_{i,i} = A_{i,i} \quad \forall i \\
& S = \text{diag}(s) \wedge s_i,\, A_{i,j} \in \{0,1\} \quad \forall i,j
\end{aligned}$$

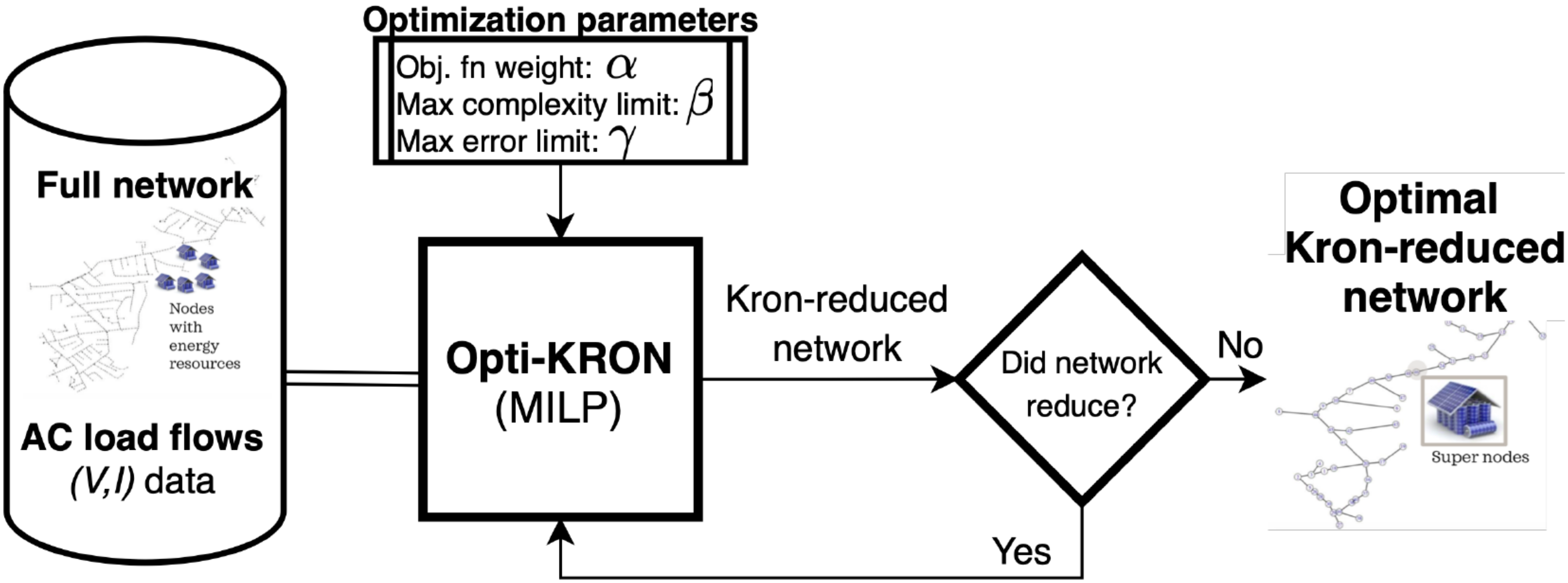
Step 1+2: Big-M on voltage error is linear

Step 3: Laplacian enforces neighbor-only Kron

Neighbor-only limits reduction! How do we extend reduction beyond just neighbors?

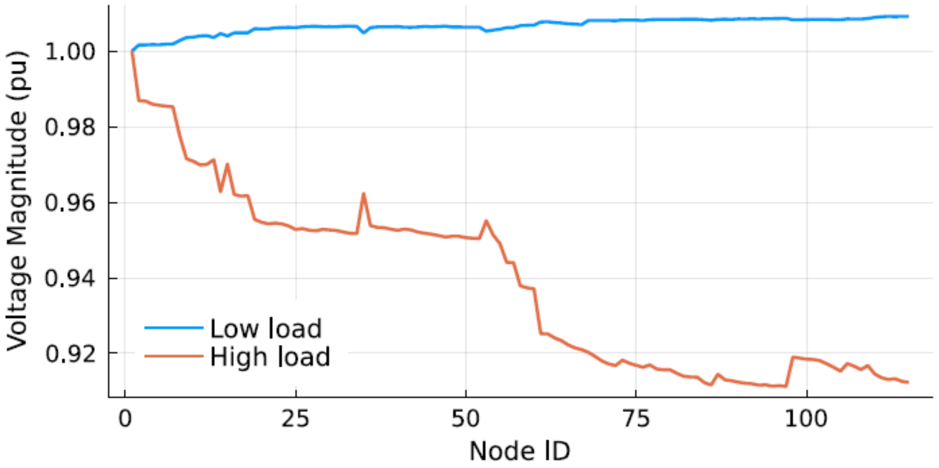
Opti-KRON: Optimal Kron-based Reduction Of Networks

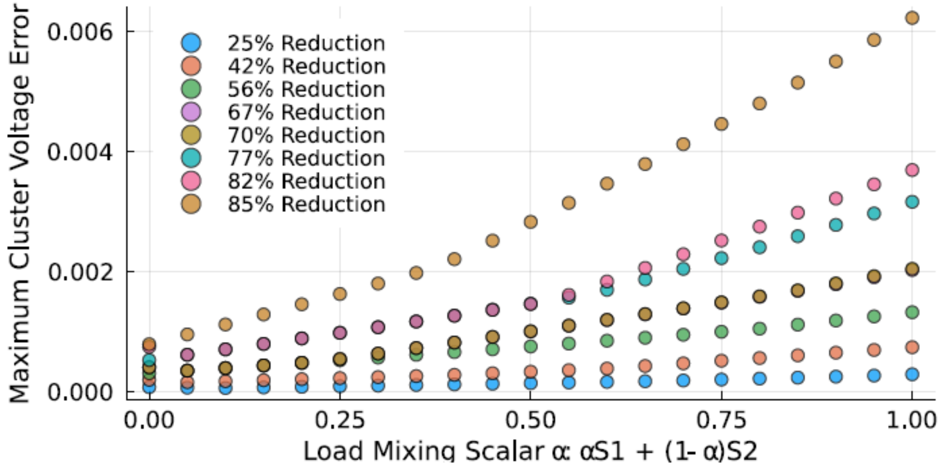
- Iterative network reduction procedure:



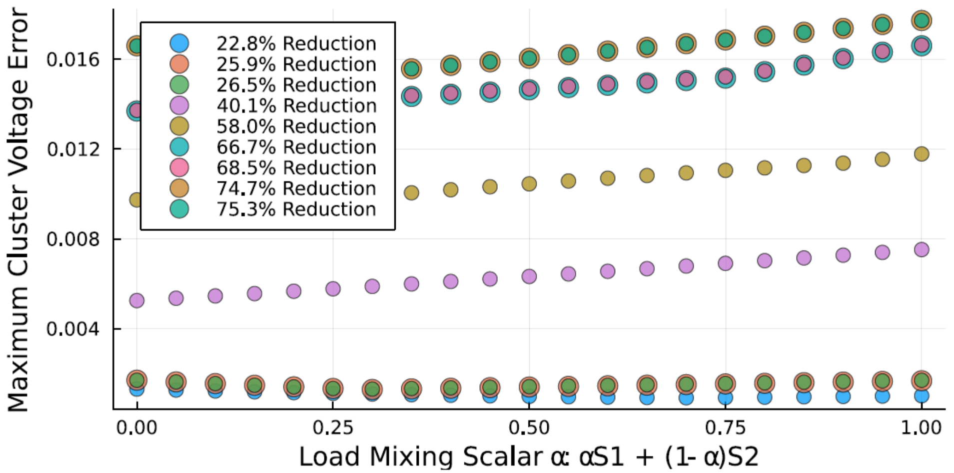
Test Results: radial and meshed networks

- Radial system (115-bus, single phase)





- Meshed network (200-bus, single phase)



Take-aways:

- Radial networks solved (0% gap) within 5 seconds
- Meshed network much slower (10's/100s seconds)
- Algorithm is sensitive to weighting parameters. Sometimes, it would make very interesting reduction decisions!

Conclusions and Future Work

We designed a Mixed-Integer Linear Program (MILP)-based tool called **Opti-KRON**

– Cast Kron-reduction as a tractable-*ish* Mixed-integer optimization problem

– Opti-KRON determines optimal network aggregation and Kron reduction for electrical networks

– Opti-KRON performance was excellent in the context of radial grids

Future work

- Improve performance on meshed grids
- Scale, test, & improve computational performance of Opti-KRON on large networks
- Consider additional metrics in the objective function
- Use Opti-KRON's reduced networks to accelerate OPF and study lifted solutions

Thank you! Questions? Comments?

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https://samchevalier.github.io/
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