Towards a Macromodel for Packetized Energy Management of Resistive Water Heaters*†

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Abstract—This paper presents a state bin transition (macro)model for a large homogeneous population of thermostatically controlled loads (TCLs). The energy use of these TCLs is coordinated with a novel bottom-up asynchronous, anonymous, and randomizing control paradigm called Packetized Energy Management (PEM). A macro-model for a population of TCLs is developed and then augmented with a timer to capture the duration and consumption of energy packets and with exit-ON/OFF dynamics to ensure consumer quality of service. PEM permits a virtual power plant (VPP) operator to interact with TCLs through a packet request mechanism. The VPP regulates the proportion of accepted packet requests to allow tight tracking of balancing signals. The developed macro-model compares well with (agent-based) micro-simulations of TCLs under PEM and can be represented by a controlled Markov chain.

Index Terms—Packetized energy management, state bin transition model, controlled Markov chain, thermostatically controlled loads, modeling.

I. Introduction

At high levels of renewable penetration, the current operating paradigm for reliably managing the variability of wind and solar generation requires having more fast-ramping conventional generators online. However, that leads to more generators idling, burning fuel, and increasing harmful air-emissions. Therefore, there is a need to move away from this traditional form of ensuring reliability to consider an active role for flexible and controllable net-load energy resources, e.g., plug-in electric vehicles (PEVs), thermostatically-controlled loads (TCLs), energy storage, and distributed generation at the consumer level [1]. Indeed, there is a growing consensus that balancing supply and demand in power systems with large amounts of variable renewable energy will require an active role for flexible distributed energy resources (DERs). While the core concepts underlying modern demand-side management (DSM) have existed for decades [2], the technology for coordinating DERs is nascent.

This paper presents preliminary analysis for a model of the aggregate system response (i.e., a *macromodel*) of a population of homogeneous TCLs operating under a novel bottom-up load coordination framework, *packetized energy management* (PEM) [3], [4]. PEM leverages the packet-based strategies from random access communication channels which have previously been applied to the distributed management of wireless sensor networks [5]. In particular, PEM may be thought of as a multi-channel, multi-receiver version of Aloha [6] or RTS/CTS

(*request/clear to send*) [7]. Under PEM, the delivery of energy to a load is accomplished using multiple "energy packets," just as digital communication networks break data into packets.

With the proposed PEM architecture, the grid operator or aggregator only requires a two-dimensional measurement from the collection of loads: aggregate power consumption and an aggregate request rates. This represents a significant advantage over aggregate model-estimator-controller state-space approaches in [8], which require an entire histogram of states from a collection of homogeneous loads to update a bin transition macro-model (similar in spirit to the one employed herein). In [8], estimation is addressed through an observer design to estimate the histogram based on aggregate power consumption; however, in some cases, the model may not be observable [9]. Recent work has extended [8] to include higher order dynamic models and end-user and compressor delay constraints [10] and stochastic dynamical performance bounds [11]. Specifically, the modeling of packet duration in this paper was inspired by the compressor lock-out method utilized in [10]. A mean-field approach to direct load control is developed for heterogenous TCL populations in [12]. Similarly to the PEM paradigm, the mean-field approach developed in [9], [13] maintain quality of service (QoS) through exit-ON/OFF mechanisms and inject randomization based on local state variables, which limits synchronization effects and promotes equitable access to the grid. In contrast to those prior works, PEM does not have the coordinator broadcast the control signal (in top-down fashion).

The most closely related work on energy packets is found in [14], where a distributed (binary information) version of a packetized load controller is proposed that also requires only (binary) packet request information from the loads. Unlike the proposed PEM scheme, [14] assumes that the exact number of participating packetized loads at any given time is known, the allocation of packet requests from the queue is synchronized, and the queue stores packet requests if the packets cannot be allocated, which creates delays in service. Instead, the work herein extends the authors' previous packetized energy results for managing PEVs [15], [16]. Specifically, this paper builds on the novel PEM paradigm for TCLs developed in [4] to present the first macro-model of TCLs under PEM. The specific contributions of this paper are three-fold:

i. Presentation of a novel macromodel that captures the mean-field response of resistive water heaters under PEM and is shown to be a controlled Markov chain.

ii. Explicit consideration of end-consumer QoS guarantees in the macro-model by including opting-out mechanics.

iii. Simulation-based validation of the PEM macro-model with agent-based micro-simulations.

II. PEM Fundamentals

PEM is illustrated in Fig. 1 and by the following events:

1. A TCL measures its local energy state (e.g., temperature).
2. If the state exceeds its limit, the TCL exits PEM, reverts to conventional hysteretic TCL control until the state is returned to within limits, and returns to Step 1. Else, based on the state, the TCL stochastically requests energy from the grid for a pre-specified epoch (an energy packet) and goes to Step 3.
3. The aggregator (or Virtual Power Plant, VPP) either accepts or denies the TCL's request, depending on system conditions, such as binding constraints or power tracking error. If denied, immediately return to Step 1. If accepted, the TCL consumes energy for an epoch and returns to Step 1.

By employing a probabilistic automata at each TCL that is capable of exiting PEM to guarantee consumer Quality of Service (QoS), we inject randomization to the load requests based on local state variables, which prevents synchronization, guarantees consumer QoS, and promotes fair access to the grid. Fig. 1 illustrates the closed-loop system under PEM.
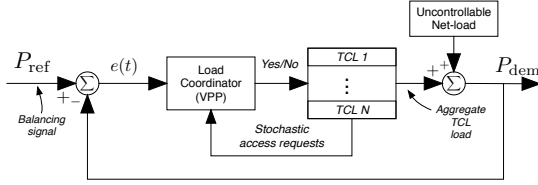

Fig. 1. Closed-loop feedback system for PEM with $P_{\text{ref}}(t)$ provided by the Grid Operator and the aggregate net-load $P_{\text{dem}}(t)$ measured by VPP.

The TCL load type of interest is electric water heaters (EWHs). The following presents a first-order single-heating-element thermodynamic model motivated by [17] but modified to consider a uniform thermal mass and hot-water withdrawal by the consumer in liters/min rather than as a fixed energy loss. Thus, consider the hybrid ON/OFF dynamics of the temperature of TCL n at time t, $T_n(t)$:

$$\dot{T}_n(t) = \frac{P_n^{\text{rate}} z_n(t)}{c\rho L_n \eta} - \frac{T_n(t) - T_{\text{amb}}}{\tau_n} - \frac{T_n(t) - T_{\text{in}}}{60 L_n} w_n(t) \quad (1)$$

where $c = 4.186$ [kJ/kg-°C] and $\rho = 0.990$ [kg/liters] represent specific heat capacity and density of water close to 50°C. L_n [liters] represents the total capacity of the EWH. The input power when ON (i.e., binary $z_n \equiv 1$) is P_n^{rate} [kW], heat-transfer efficiency η is assumed as 1.0, ambient losses are described by time-constant τ_n, and the uncontrolled hot-water withdrawal rate (i.e., noise) is w_n [liters/min]. The terms T_{amb} and T_{in} are, respectively, the ambient and inlet temperatures [°C], which are considered constant in this paper.

In the discrete-time implementation of PEM, the probability that TCL n with local temperature $T_n[k]$ and (homogeneous) dead-band $[T_{\text{min}}, T_{\text{max}}]$ requests access to the grid during time-step k (over interval Δt) is defined by the cumulative exponential distribution function $P(T_n[k]) := 1 - e^{-\mu(T_n[k])\Delta t}$, where rate parameter $\mu(T_n[k]) > 0$ is dependent on the local temperature. This dependence is explicitly stated by the following PEM rate parameter:

$$\mu(T_n[k]) = \begin{cases} 0, & \text{if } T_n[k] > T_{\text{max}}, \\ \frac{T_{\text{max}} - T_n[k]}{T_n[k] - T_{\text{min}}} m_R, & \text{if } T_n[k] \in (T_{\text{min}}, T_{\text{max}}], \\ \infty, & \text{if } T_n[k] \leq T_{\text{min}}, \end{cases} \quad (2)$$

where $m_R > 0$ [Hz] is a design parameter that defines the mean-time-to-request (MTTR). For example, if one desires a MTTR of 5 minutes at $T_n^{\text{set}} := \frac{T_{\text{min}} + T_{\text{max}}}{2}$ then $m_R = \frac{1}{600}$Hz.

Within the PEM scheme displayed in Fig. 1, the VPP tracks a balancing signal by responding to individual downstream asynchronous and stochastic load access packet requests with "YES" or "NO" notifications based on the output error between actual aggregate output, $P_{\text{dem}}(t)$, and the VPP's reference signal, $P_{\text{ref}}(t)$: $e(t) := P_{\text{ref}}(t) - P_{\text{dem}}(t)$. The VPP is similar to a relay controller in the sense that it accepts a request ("YES") if $e(t) > 0$, otherwise, "NO." However, unlike standard relay control of a single plant, the VPP responds to asynchronous requests from many plants, which overcomes common drawbacks associated with relay control (e.g., switching causing oscillations) and allows accurate tracking.

Unlike the authors' prior heterogeneous micro-simulation work presented in [4], this paper will focus on a homogeneous population of EWHs with constant parameters. Upon aggregation of thousands of EWHs, state augmentation is intractable. Therefore, we seek to develop a suitable state bin transition macro-model for the homogeneous population.

III. Analysis of Hot-water Withdrawal Rates

In this section, we describe the stochastic modeling of the uncontrolled hot-water withdrawals, $w_n(t)$. To clarify notation, we will omit subscript n as this section focuses on a single TCL. Assume that there exists an appropriate probability space $(\Omega, P, \mathcal{F})$, where Ω is the set of events, $\mathcal{F}$ a filtration, and P the probability measure of elements in $\mathcal{F}$. For this purpose, a *Poisson rectangular pulse* stochastic differential model is employed [18]. That is,

$$dw(t) = (v(t) - w(t))\, dN_1(t) - w(t)\, dN_2(t), \quad (3)$$

where $v(t)$ is an exponentially distributed random variable with mean λ representing the intensity of hot-water withdrawals for all t and N_1, N_2 are independent stationary Poisson point processes with constant rate parameters λ_1, λ_2, respectively, representing the distribution of the duration of hot-water withdrawal events. Thus, it follows that the random variable N_1 initiates a hot-water event (i.e., increases the withdrawal rate) while N_2 ends a hot-water event (i.e., decreases the withdrawal rate).

Computing the average TCL water consumption profile given λ, λ_1 and λ_2 (for the entire population) is critical for studying the aggregated behavior. This is made more clear in Section IV when the transition probabilities are computed for an aggregated system of electric water heaters in steady state. Denote the expected value of a random process by $\bar{x}(t) := E[x(t)]$. Due to the independence of the processes ΔN_1, ΔN_2 and v in time, one can compute the expected water consumption for each TCL as

$$\frac{d\bar{w}(t)}{dt} = (\bar{v}(t) - \bar{w}(t))\lambda_1 + \bar{w}(t)\lambda_2. \quad (4)$$

Note that $\bar{v}(t)$ is constant since its distribution is known for all times. The solution of (4) when $w(0) = 0$ is

$$\bar{w}(t) = \bar{v}\frac{\lambda_1}{\lambda_1 + \lambda_2}(1 - \exp(-(\lambda_1 + \lambda_2)t))$$

The expected water draw reaches steady state as t goes to infinity. Hence, the steady state water consumption is

$$\bar{w}_{\text{sst}} := \lim_{t \to \infty} \bar{w}(t) = \bar{v} \frac{\lambda_1}{\lambda_1 + \lambda_2}. \tag{5}$$

The constant value $\bar{w}_{\text{sst}}$ computed in (5) is used in Section IV to estimate the state transition probabilities of the aggregate EWH temperatures from (1).

IV. PEM CONTROLLED MARKOV MODEL

A macro-model for a large population of TCLs is developed in this section as an abstraction of the augmented (agent-based) dynamic micro-models. Specifically, consider the TCL population dynamics over a discretization of the temperature state space and employ a state bin transition model, such that the macro-model approaches the behavior of the micro-model as the number of devices increases [19]. The transitions between these bins are determined by the dynamical system equations of the homogeneous TCLs as discussed below.

A. Conventional Thermostatic Control

The macro-model utilizes a finite set $\mathcal{X} = \{x_1, \dots, x_N\}$, where each element is called a state. Recall the probability space $(\Omega, \mathcal{F}, P)$ introduced in Section III. Then, random variables $\{X_k\}_{k \geq 0}$ are defined as $X_k : \Omega \to \mathcal{X}$. Let $x_j \in \mathcal{X}$ and denote $q_j[k] = P(X_k = x_j)$ as the probability of $X_k = x_j$, $k \geq 0$. The column vector $q[k] := (q_1, \dots, q_N)^T$ then gives the probability mass function of the random variable X_k. Also, if one denotes the transition probability of an homogeneous Markov chain as $p_{ij} = P(X_{k+1} = x_i | X_k = x_j)$, it then follows that

$$q[k+1] = Mq[k], \tag{6}$$

where $M = \{p_{ij}\}_{1 \leq i,j \leq N}$ [20]. Given an initial distribution $q[0]$, one can solve for (6) and find the distribution at time k as $q[k] = M^k q[0]$.

Conventional thermostatic control of an EWH is based on keeping the local state variable (e.g., temperature) within a deadband $[T_{min}, T_{max}]$ by switching the device ON (when $T \leq T_{min}$) or OFF (when $T \geq T_{max}$). More precisely, the interval $[T_{min}, T_{max}]$ is divided into N consecutive bins. Since (1) includes hybrid ON/OFF dynamics, the state space for the system consists of two discrete state spaces: $\mathcal{X}_{\text{on}}$ and $\mathcal{X}_{\text{off}}$. That is, the full state space is given by $\mathcal{X} = \mathcal{X}_{\text{on}} \cup \mathcal{X}_{\text{off}}$. At time k, the probability mass function of the system is $q^\top = (q_{\text{on}}^\top, q_{\text{off}}^\top)$ with $q_{\text{on}} = (q_{\text{on}}^1, \cdots, q_{\text{on}}^N)^\top$ and $q_{\text{off}} = (q_{\text{off}}^1, \cdots, q_{\text{off}}^N)^\top$. Note that q contains the percentage of the population in each state of $\mathcal{X}$. For instance, if R is the total number of EWHs and R_{on}^i is the number in state x_{on}^i, then $R_{\text{on}}^i = q_{\text{on}}^i R$. Similarly, the total ON-population is given by

$$y = c^\top q \quad \text{for} \quad c = (\mathbf{1}_N^\top, 0 \cdots 0)^\top \in \mathbb{R}^{2N}, \tag{7}$$

and $\mathbf{1}_N = (1, \dots, 1)^\top \in \mathbb{R}^N$. The transition rates are computed by considering how the temperature bin corresponding to a particular state is altered by the hybrid dynamics in (1).

Together with discrete sampling time and temperature bin widths, the hot water withdrawal rate w, described in Section III, is one of the main factors affecting these transition rates. For example, consider two realizations a, b of the water profiles generated by (3) with identical parameters except for the water withdrawal intensities of the random variable v
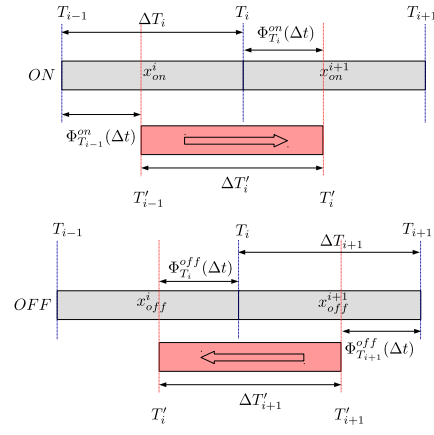

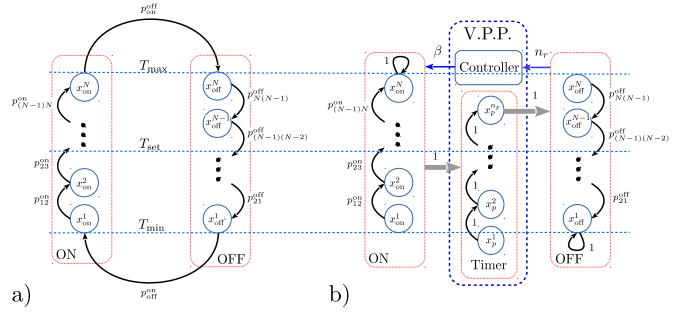Fig. 2. Transition rates calculation for *on* and *off* populations.

Fig. 3. Abstraction for (a) conventional thermostatic control and (b) PEM control, where self-loops are not visualized.

($\lambda_a \neq \lambda_b$). An ON TCL with λ_a ($< \lambda_b$) at temperature T_i reaches temperature T_{i+1} faster than realization b, which draws more hot water on average and increases the time required to reach T_{i+1}. Nevertheless, since the hot water draw profiles in the population are assumed to be statistically identical, the average of these profiles reaches steady state (5) when $t \to \infty$. Thus, the state transition rates for the large population are calculated considering the evolution of (1) with respect to the average hot water draw of the population. The transition rates for the ON and OFF populations are computed next. Dropping the subscript n in (1), it follows that the solution with steady state consumption $w = \bar{w}_{\text{sst}}$ and $T(0) = T_0$ is

$$T(t) = \Phi_{T_0}(t) = e^{-at}\left(T_0 - \frac{b(z)}{a}\right) + \frac{b(z)}{a}, \tag{8}$$

where $a = \frac{1}{\tau} + \frac{\bar{w}_{\text{sst}}}{60L}$ and $b(z) = \frac{T_{\text{amb}}}{\tau} + \frac{T_{\text{in}}}{60L}\bar{w}_{\text{sst}} + \frac{P^{\text{rate}}}{c\rho L \eta}z$. In particular, define $\Phi_{T_0}^{\text{on}}(t) = \Phi_{T_0}(t)\big|_{z=1}$ and $\Phi_{T_0}^{\text{off}}(t) = \Phi_{T_0}(t)\big|_{z=0}$. For the ON population, the dynamics imply forward transitions, i.e., from x_{on}^i to x_{on}^{i+1} as shown in Fig. 2. First, take the boundaries of the temperature bin T_{i-1} and T_i corresponding to state x_{on}^i and compute $T_{i-1}' = \Phi_{\Delta t}^{\text{on}}(T_{i-1})$ and $T_i' = \Phi_{\Delta t}^{\text{on}}(T_i)$. Note that in this case $T_i < T_i'$. Thus, the percentage of water heaters that remain in x_{on}^i and move to x_{on}^{i+1}, respectively, are given by

$$p_{ii}^{\text{on}} = \frac{T_i - T_{i-1}'}{T_i' - T_{i-1}'} \quad \text{and} \quad p_{i(i+1)}^{\text{on}} = \frac{T_i' - T_i}{T_i' - T_{i-1}'}.$$

Note that $p_{ii}^{\text{on}} + p_{i(i+1)}^{\text{on}} = 1$, as expected. Transition rates for

the OFF dynamics are determined similarly, but are reversed, i.e., from x_{off}^{i+1} to x_{off}^i since $T_i' = \Phi_{\Delta t}^{\text{off}}(T_i) < T_i$. Thus,

$$p_{ii}^{\text{off}} = \frac{T_{i+1}' - T_i}{T_{i+1}' - T_i'} \quad \text{and} \quad p_{i(i-1)}^{\text{off}} = \frac{T_i - T_i'}{T_{i+1}' - T_i'}.$$

The previous analysis was purposely restricted to state transitions between contiguous states. Using (8), one can compute an upper bound for Δt such that any EWH in state x_{on}^i only transitions to x_{on}^{i+1} and any EWH in x_{off}^{i+1} only transition to x_{off}^i for all i. Define

$$t_i^{\text{on}} = a^{-1} \log \left(\frac{T_i - \frac{b(z)}{a}}{T_{i+1} - \frac{b(z)}{a}} \right) \Bigg|_{z=1} \tag{9}$$

as the time that an EWH takes to go from T_i to T_{i+1}. Observe that if an EWH at T_i is kept ON for $t > t_i^{\text{on}}$ seconds, then $T(t) > T_{i+1}$. This implies that some EWHs in x_{on}^i will transition to x_{on}^{i+2} and skip x_{on}^{i+1}. Similarly, t_i^{off} is defined as in (9) but $z = 0$ and the transitions are in a reverse direction. The condition on the discretization time step Δt for *contiguous transitions* is then formulated as $\Delta t < \min_i\{t_i^{\text{on}}, t_i^{\text{off}}\}$. For instance, a choice of $N = 30$ for the simulation in this paper means that $\Delta t < 60.27$ seconds.

In addition, the OFF-to-ON ($p_{\text{off}}^{\text{on}}$) and ON-to-OFF ($p_{\text{on}}^{\text{off}}$) transition rates must be computed to capture the jump to a transitory state that automatically transitions to x_{on}^1 and x_{off}^N, respectively. The complete Markov chain for conventional thermostatic control is shown in Fig. 3a. It is important to observe that the transient effects on temperature caused by stochastic hot water withdrawals are not captured since the transition rates assume a steady state (mean) consumption of hot water. The Markov transition matrix M associated to conventional thermostatic control is then given as

$$M = \begin{pmatrix} p_{11}^{\text{on}} & 0 & \cdots & 0 & p_{\text{on}}^{\text{off}} & \cdots & 0 & 0 \\ p_{12}^{\text{on}} & p_{22}^{\text{on}} & \ddots & 0 & 0 & \cdots & 0 & 0 \\ 0 & p_{23}^{\text{on}} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & p_{NN}^{\text{on}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & & 0 & p_{11}^{\text{off}} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \ddots & p_{(N-1)(N-2)}^{\text{off}} & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & p_{(N-1)(N-1)}^{\text{off}} & p_{N(N-1)}^{\text{off}} \\ 0 & 0 & \cdots & p_{\text{off}}^{\text{on}} & 0 & \cdots & 0 & p_{NN}^{\text{off}} \end{pmatrix}. \tag{10}$$

Observe that the Markov chain associated to M is irreducible since one can reach any state from any arbitrarily initial state. It follows then that this abstraction possess a unique invariant distribution as well since $\mathcal{X}$ is finite dimensional. Nonetheless the conventional model lacks the flexibility inherent to PEM.

B. PEM Markov model

Recall from Section II that, under PEM, an EWH can only switch ON for an epoch if its packet request is accepted by the VPP coordinator. That is, given the aggregate request rate, the VPP selects the proportion of EWHs that will receive a packet and switch ON for each time-step. To capture the unique nature of PEM's fixed packet duration and VPP's role, we leverage prior literature on fault tolerant recovery logic [21] and TCL modeling with compressor lockout periods [10]. Specifically, a

fixed *timer* is added to the state bin transition model to track the population with accepted packet requests. The objective of this section is to present a macro-model describing PEM control as a *controlled Markov chain*.

Definition 1 (Controlled Markov Chain from [20]): Let $\{u_k\}_{k\geq 0}$ be a sequence of real valued functions taking values on a set U. A Markov chain $\{X_k\}_{k\geq 0}$ is said to be a *controlled Markov chain* (CMC) if its transition matrix $M = M(u) := \{q_{ij}(u)\}_{1\leq i,j\leq N}$ satisfies

$$P(X_{n+1} = x_{i_{n+1}} | X_n = x_{i_n}, \ldots, X_0 = x_{i_0}, u_n, \cdots, u_0)$$
$$= P(X_{n+1} = x_{i_{n+1}} | X_n = x_{i_n}, u_n) = p_{i_{n+1}i_n}(u_n).$$

The definition also implies that $M(u)$ for a CMC must be a (column) stochastic matrix for any choice of $u \in U$. Assuming that all states of the CMC are observed, one can define a control policy: $u = \mathcal{X} \to U$, and, thus, $M = M(u(x))$. The probability mass function of a CMC is computed similarly using $q[k+1] = M(u[k])q[k]$ given an initial distribution $q[0]$ and control input $u[0]$.

In what follows, PEM control will be introduced in the context of CMCs. The underlying Markov transition matrix over which PEM is implemented is given by (10), but with $p_{\text{off}}^{\text{on}} = p_{\text{on}}^{\text{off}} = 0$ and $p_{NN}^{\text{on}} = p_{11}^{\text{off}} = 1$. That is, x_{off}^1 and x_{on}^N are absorbent states indicating that ON states can not be reached from OFF states and vice-versa. VPP control, therefore, becomes the interface between these two modes of operation. The mechanics of switching EWHs ON and OFF under PEM control are described next.

Suppose $q[k] \in \mathbb{R}^{2N}$ is the probability distribution of the PEM macro-model population at time k, $\beta_{\text{on}} = \text{diag}\{\beta_{\text{on}}^1, \ldots, \beta_{\text{on}}^N\}$ with $\beta_{\text{on}}^i \in [0,1]$ the percentage of the OFF-population in state x_{off}^i that is switched ON by the VPP, and $\beta_{\text{off}} = \text{diag}\{\beta_{\text{off}}^1, \ldots, \beta_{\text{off}}^N\}$ with $\beta_{\text{off}}^i \in [0,1]$ the percentage of the ON-population in state x_{on}^i that is switched OFF. The action of switching ON and OFF on q is given by the transformation:

$$\bar{q}[k] = \bar{M}(\beta_{\text{on}}[k], \beta_{\text{off}}[k]) \, q[k], \tag{11}$$

where

$$\bar{M}(\beta_{\text{on}}, \beta_{\text{off}}) = \left(\begin{array}{c|c} I_N - \beta_{\text{off}} & \beta_{\text{on}} \\ \hline \beta_{\text{off}} & I_N - \beta_{\text{on}} \end{array} \right), \tag{12}$$

and I_N denotes the N-dimensional identity matrix. Once $\bar{M}(\beta_{\text{on}}, \beta_{\text{off}})$ has switched some EWHs ON and some other OFF, the underlying transition matrix M acts on $\bar{q}$. This provides the dynamics

$$q[k+1] = M\bar{M}(\beta_{\text{on}}, \beta_{\text{off}})q[k]. \tag{13}$$

The next theorem simply says that the sequence $\{X_k\}_{k\geq 0}$ associated (13) is a CMC.

Theorem 1: Let $\beta_{\text{on}}[k], \beta_{\text{off}}[k] \in \mathbb{R}^{N\times N}$ be defined as in (11) for all $k \geq 0$. The sequence $\{X_k\}_{k\geq 0}$ of random variables X_k taking values in $\mathcal{X}$ and probability distribution satisfying (13) is a controlled Markov chain as described by Definition 1 with input $u[k] = (\mathbf{1}_N^\top \beta_{\text{on}}[k], \mathbf{1}_N^\top \beta_{\text{off}}[k])^\top \in \mathbb{R}^{2N}$.

Proof: The proof is straightforward since (10) and (12) are stochastic matrices for arbitrary $\beta_{\text{on}}^i, \beta_{\text{off}}^i \in [0,1]$ for all i, and the product of stochastic matrices is a stochastic matrix. $\blacksquare$

An important aspect of PEM control is that only EWHs that

are OFF request a packet and do so as a function of the (local temperature) bin, which implies that not all OFF EWHs will turn ON. Therefore, define

$$\hat{q}[k] := \hat{M}q[k] = \left(\begin{array}{c|c} I_N & 0_N \\ \hline 0_N & T_{req} \end{array}\right) q[k] = \left(\begin{array}{c} q_{\text{on}}[k] \\ \hat{q}_{\text{off}}[k] \end{array}\right),$$

where $T_{req} = \text{diag}\{p_1^{req}, \ldots, p_N^{req}\}$ and $p_i^{req} := 1 - e^{-\mu(T_i^m)\Delta t}$ is the request probability assigned to x_{off}^i by (2) with respect to the mid-point of temperature bin i, T_i^m. Note that $\hat{q}$ is not a probability mass function since $\mathbf{1}_N^\top(q_{\text{on}} + \hat{q}_{\text{off}}) < 1$. The aggregate request rate, i.e., the population that can be switched ON, is now given by:

$$n_r[k] := \mathbf{1}_N^\top \hat{q}_{\text{off}}[k]. \tag{14}$$

Under PEM, the VPP determines the rate of accepting packets, $\beta[k]$. The resulting EWHs instantly switch ON when packet requests are accepted. The population of devices that switch from OFF to ON, q^+, is a function of β and q_{off}. That is,

$$q^+[k] := \left(\begin{array}{c|c} 0_N & \beta[k]T_{\text{reg}} \\ \hline 0_N & -\beta[k]T_{\text{reg}} \end{array}\right) q[k] = M_{\beta[k]}^+ q[k] \tag{15}$$

In contrast, to model the population of EWHs that switch from ON to OFF requires information on the rate of expiring packets. In other words, let δ [secs] be the duration of a packet epoch, then the EWHs that have been ON for δ seconds will turn OFF. This requires keeping track of how many EWHs were turned ON δ seconds ago and, essentially, constitutes a delayed system. However, one can augment states to the system dynamics to account for the needed memory, which is equivalent to having a timer. That is, given δ, the time step Δt, and the vector of augmented (timer) states $x_p \in \mathbb{R}^{n_p}$ with $n_p = \lfloor \delta/\Delta t \rfloor$, the *timer dynamics* is given by

$$x_p[k+1] = M_p x_p[k] + C_p q_{\text{on}}^+[k] \text{ and } y_p[k] = x_p[k], \tag{16}$$

where $M_p \in \mathbb{R}^{n_p \times n_p}$ is a zero matrix except for its first lower diagonal whose components are 1 and $C_p \in \mathbb{R}^{n_p \times N}$ is responsible for allocating the newly switched ON population into the timer states. Note that there exists a temperature T_p such that $\Phi_{T_p}^{\text{on}}(\delta) = T_{max}$. Therefore it is necessary for C_p to interrupt packets to prevent exceeding temperature limits. Specifically, if $T_{i+1} < T_p$, C_p allocates all requesting EWHs from bin $[T_i, T_{i+1}]$ into the timer state x_p^1. Otherwise, it allocates EWHs with $T_j > T_p$ in the timer state x_p^j with $j = \lfloor (\delta - t_j)/\Delta t \rfloor$ and t_j is the time it will take to increase the EWH's temperature from T_j to T_{max}. Note that since the macromodel considers only binned (rather than exact) temperatures, the allocation of requests assumes that the mass function in each state is uniformly distributed.

Observe that the timer states are internal to and inform the VPP of the distribution of total ON population in PEM (i.e., $1_N^\top q_{\text{on}}$) across all packet intervals, x_p. As in (15), one can define the population of EWHs that just completed their δ-second packet and will turn OFF instantly as

$$q^-[k] := \left(\begin{array}{c|c} \beta^-[k]I_N & 0_N \\ \hline -\beta^-[k]I_N & 0_N \end{array}\right) q[k] = M_{\beta^-[k]}q[k], \tag{17}$$

where $\beta^-[k] := y_p^{n_p}[k]/(\mathbf{1}_{n_p}^\top y_p[k])$. One can now formulate

the ON/OFF switching events for the entire population as:

$$\bar{q}[k] := q[k] + q^+[k] - q^-[k] = (I + M_{\beta[k]}^+ - M_{\beta^-[k]}^-)q[k],$$

which yields the EWH population dynamics:

$$q[k+1] = M(I + M_{\beta[k]}^+ - M_{\beta^-[k]}^-)q[k]$$
$$= \bar{M}(\beta_{\text{on}}[k], \beta_{\text{off}}[k])q[k], \tag{18}$$

where $\beta_{\text{on}}[k] = \beta[k]T_{req}$ and $\beta_{\text{off}}[k] = \beta^-[k]I_N$. Note that there is no order in which EWHs are switched ON or OFF since both happen simultaneously. Fig. 3b shows the state diagram of the population model under PEM control.

The next corollary follows directly from Theorem 1.

Corollary 1: The sequence $\{X_k\}_{k \geq 0}$ of random variables X_k taking values in $\mathcal{X}$ and probability distribution satisfying (18) is a controlled Markov chain with input $u[k] = (\mathbf{1}_N^\top \beta[k]T_{req}, \mathbf{1}_N^\top \beta^-[k]I_N)^\top$.
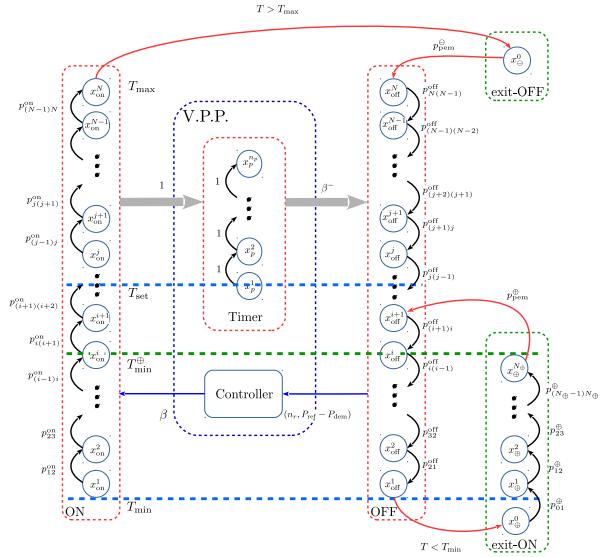

Fig. 4. PEM macromodel with exit-ON ($\oplus$) dynamics. ON/OFF state transitions are controlled by VPP and illustrated with gray and blue arrows.

C. Tracking with PEM Macromodel

In PEM, the input β is exogenous. Recall P_{avg}, P_{ref} and P_{dem} (see [4] for a list of the system parameters values) denote the average, reference and demand power for the large scale water heater system. Given n_r in (14) and that PEM tracking is activated (per Fig. 1), the input $\beta[k]$ in Fig. 3b is generated by the VPP at each instant of time k as

$$\beta[k] = \min\left\{1, \frac{P_{\text{ref}}[k] - P_{\text{dem}}[k]}{P_{\text{avg}} n_r[k]}\right\}$$

when $P_{\text{ref}} > P_{\text{dem}}$ and 0, otherwise. Observe in the diagram that the timer dynamics automatically releases the population in $x_p^{n_p}$ and transitions them all to the OFF states. Also, note that if $\beta[k] = 0$ for all k then the state diagram becomes reducible since there the states cannot transition from ON to OFF. This last fact is undesirable given that x_{off}^1 ends up accumulating the entire population when k goes to infinity, which implies that every water heater becomes synchronized. This short-coming is addressed by additional states that will allow cold EWHs to turn ON even when the VPP sets $\beta[k] = 0$. This *exit*-ON/OFF mechanism is augmented to the PEM macro-model to ensure QoS as described next.

D. Exit-ON/OFF Dynamics

End-consumer QoS is of paramount importance when controlling a large scale system of TCLs. Specifically, *no one likes to take a cold shower*. Therefore, whenever an EWH's temperature falls outside the dead-band $[T_{\min}, T_{\max}]$, it will exit the packetized scheme and revert to conventional control until a pre-specified PEM opt-in set-point is reached. Once the opt-in set-point is reached, the EWH is allowed to re-enter the packetized scheme.

The population of EWHs that are too cold and exit PEM (to turn ON) join the *exit*-ON mode dynamics (denoted by $\oplus$). On the other hand if a water heater is too hot and has to turn OFF, then it joins *exit*-OFF mode dynamics (denoted by $\ominus$) at state $x^0_\ominus$ after which EWHs transition under M naturally to the requesting states. These two PEM exit modes of operations were introduced in Section II as they appeared in [4]. Adding these modes of operation to the PEM macro-model only requires a simple augmentation of states with their corresponding transition rates as shown in Fig. 4. The updated full population dynamics is given by (16) and

$$q[k+1] = M_{\text{exit}}(I + M^+_{\beta[k]} + M^-_{\beta^-[k]})q[k], \quad y[k] = c^\top q[k],$$

where $M_{\text{exit}} := \text{diag}\{M_{\text{exit-ON}}, \bar{M}, M_{\text{exit-OFF}}\}$, $M_{\text{exit-ON}}$ is a matrix of zeros except for the main diagonal $(p^\oplus_{11}, \ldots, p^\oplus_{N_\oplus N_\oplus})$ and the first lower diagonal $(p^\oplus_{12}, p^\oplus_{23} \ldots, p^\oplus_{(N_\oplus-1)N_\oplus})$, $M_{\text{exit-OFF}}$ introduces the probabilities to re-enter PEM from x^N_{on} to $x^0_\ominus$ and from $x^0_\ominus$ to x^N_{off} with $p^\ominus_{\text{pem\_}}$ corresponding to the exit-OFF mode, and $\bar{M}$ is such that $\bar{M}_{ij} = M_{ij}$ except for $M_{(N+N_\oplus+1)N_\oplus} = p^\oplus_{\text{pem}}$, which describe the transition probabilities to re-enter PEM from the exit-ON mode.

E. Simulation Results

Finally, the micro and macro simulations are compared against each other. Fig. 5 shows a 16 hour simulation for both the micro and macro models. Both simulations have been accepting all requests ($\beta = 1$) long enough to show their steady state and then PEM tracking was switched on with five minute packets. The chosen time-step for both simulations was $\Delta t = 5$ seconds. It can be observed that since the steady state (no tracking is been performed) of both simulations agree on average. Also in this simulation, the tracking error remains within $\pm 5\%$.
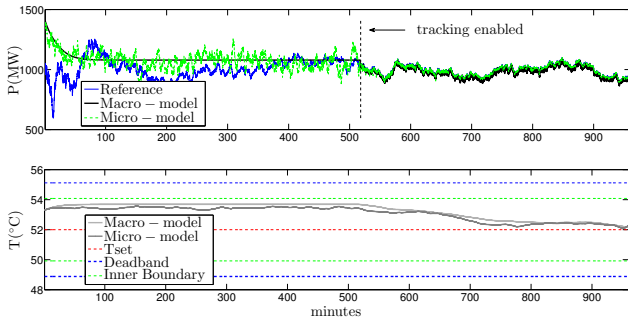

Fig. 5. PEM control tracking.

V. Conclusions

A macro-model for PEM control was developed in terms of a controlled Markov chain. Quality of service was considered by introducing exit-ON/OFF modes of operation. Future work involves further formalization of the macro-model and explore different approaches to incorporate population heterogeneity.

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