Efficient Distribution System Optimal Power Flow with Discrete Control of Load Tap Changers

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***Abstract*—Load tap changers (LTCs) are commonly used for voltage control in distribution systems. The mechanical relays on LTCs are controlled in discrete steps; hence, distribution optimal power flow (DOPF) formulations require LTCs to be modeled with integer variables. Integer variables are generally ignored or relaxed and rounded off to reduce the computational burden in DOPF models. However, in this work, we highlight and overcome complex infeasibility issues caused by rounding methods. Moreover, recent advances in convex optimization have improved the computational tractability of the DOPF problem. In this work, we develop and analyze numerical efficiency of different, but exact formulations for incorporating LTCs as integer control assets in a relaxed second-order cone program (SOCP) version of DOPF model. The resulting formulation becomes a mixed-integer SOCP (MISOCP), which is computationally tractable compared to mixed-integer non-linear program (MINLP). To recover the exact optimal solution in the proposed MISOCP-DOPF model, a McCormick relaxation is employed within a sequential bound-tightening algorithm. We show that solutions obtained from MISOCP-DOPF are always AC feasible. Thus, the MISOCP-DOPF yields optimal and realistic AC solutions, and is validated in large distribution feeders and is compared to MINLP counterpart.**

***Index Terms*—Optimal power flow, mixed-integer second-order conic programming (MISOCP), McCormick envelopes, Distribution systems, Load tap changers (LTCs).**

I. Introduction

Recently, optimal power flow (OPF) problems for three-phase unbalanced distribution systems have gained significant interest. Distribution optimal power flow (DOPF) is becoming central to operate an Advanced Distribution Management System (ADMS) for applications like volt/var optimization, network reconfiguration, and distributed energy resource (DER) coordination or Distributed Energy Resource Management Systems (DERMs) [1], [2]. Conventionally, discrete controllable assets, such as mechanical load tap changers (LTCs) and capacitor banks that can only take discrete (integer) states, e.g., binary ON/OFF (cap banks) or integer states, e.g., $-16, -15, \ldots, 16$ (LTCs), have been the primary controllable elements in distribution systems used for managing reactive power and voltage limits (volt/var control) [3]. However, with increasing penetrations of behind-the-meter variable DERs (e.g., roof-top solar PV, electric vehicles, and distributed storage), the voltage regulation problem is becoming increasingly dynamic in nature. That is, managing variable voltages within prescribed limits with only conventional assets requires frequent actuation of discrete assets, which reduces the lifetime of resources [4]. To balance the use of these discrete elements with flexible and continuously controllable assets, such as DERs and advanced inverters [5], [6], optimized coordination of discrete and controllable resources is valuable and needed. However, with mixed controllable assets residing in three-phase nonlinear systems, the DOPF problem is a challenging problem to solve.

Since the LTCs can only be controlled in discrete integer steps, this turns a tractable and continuous optimization problem into a mixed-integer program (MIP), which is NP-hard[1]. Hence, large-scale MIP problems are not tractable unless certain conditions hold, such as having totally unimodular matrices [7]. For mixed-integer non-linear programming (MINLP) formulations of the DOPF problem, there is additional complexity that arises from the non-convexity of non-linear power flow equations. Non-convex MINLP and mixed-integer quadratic-constraint programming (MIQCP) models are developed in [8] and solved for a system with large number of discrete variables. However, the solutions in [8] are not guaranteed optimal, and the optimality gap is not evaluated to ensure quality of the solutions. Thus, obtaining an optimal solution to the non-convex mixed-integer DOPF problem in practical time has been recognized as a significant challenge [9]–[11].

Authors in [11]–[16] have proposed numerous techniques to solve the OPF problems with integer variables. In [11], penalty term and rounding function are used to drive relaxed solution to integer solution. In [12], the authors used the sensitivity of the objective functions to the inequality constraints for the discrete variables. An algorithm is proposed in [13] that combines ordinal optimization with distributed asynchronous dual formulations. The approach reduces the solution space to quickly find good, but sub-optimal feasible solutions. An annealing-based heuristic method is used in [14]. The performance of the algorithm is improved by approximating the stochastic simulated annealing with a set of deterministic equations; however, the method suffers from slow convergance and does not guarantee optimality. In [15], discrete variables are modeled efficiently by using a penalty function that makes the problem continuous and differentiable. However, the use of a penalty function requires careful analysis to avoid sub-optimal solutions. In [16], a hybrid approach combines primal-dual interior point and meta-heuristics to speed up the

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[1]NP-hard refers to problems where the computational complexity associated with the MIP integers grows exponentially with the number of discrete variables in the DOPF problem (i.e., no guarantee of the existence of a polynomial-time algorithm).

MINLP computation. However, this method also suffers from scalability issue due the MINLP formulation.

Thus, given the computational complexity of a MIP-based DOPF [17], state-of-the-art methods in literature approach the problem by replacing the discrete variables with continuous ones, and then project the solution onto the original discrete lattice (i.e., use rounding). However, the rounding approaches neglects the effect of discretization, which can lead to two undesirable outcomes: *(a)* infeasibility in the solution to the original MINLP model, and *(b)* sub-optimal solutions. In Section II-A, it is demonstrated with a simple 4-node feeder that simple rounding heuristics cannot guarantee feasibility. Even use of local search heuristic to ensure feasibility with rounding does not scale up due to the exhaustive local search [11]. This paper extends these ideas to a computationally tractable and scalable approach for DOPF with LTCs.

In general, reformulations are used to efficiently solve the nonlinear DOPF problem. There are two general reformulation approaches: $a)$ linear approximations and $b)$ convex relaxations. A linear approximation of the nonlinear DOPF model can involve Jacobian linearizations and/or simplifying assumptions (e.g., "DC" optimal power flow assumptions) and results in a convex optimization problem with (piece-wise) linear constraints (e.g., a linear or quadratic program, LP or QP). With a linear approximation, one can readily exploit the efficiency of LP/QP solvers. As such, linear approximations of OPF are indispensable in the industry and are widely studied [18]–[20]. Linear approximations of DOPF problem have also been widely studied [21], [22].

Convex relaxations also permit the use of polynomial-time algorithms for solving the DOPF problem [23], and provides a rich set of optimization techniques that preserve the (globally) exact model [24]–[30]. For example, semi-definite programming (SDP) [24], [29] and second-order cone programming (SOCP) [25]–[28] can recover globally optimal solutions to the non-convex DOPF problem, under certain circumstances. However, SDPs are computationally much more expensive compared to SOCPs in solving DOPF [31]. For single-phase equivalent radial networks, it has been proven that SDP and SOCP relaxations exactly recover the global optimum of the nonlinear DOPF problem [23] under certain assumptions on the objective function (e.g., loss minimization).

These relaxations have also been successfully applied to solve mixed-integer program (MIP) variants of the DOPF problem. In [32], [33], the DOPF problem was initially reformulated with integer variables. However, the convex MIP was then further relaxed by relaxing the integrality of the discrete variables. The resulting problem was then solved and rounding was used to recover an integer solution. However, as presented in the Motivating Example in Section II-A, rounding can lead to infeasibility when operating the system close to its limits. In [34], an SOCP relaxation is used for solving an DOPF problem with discrete control variables. However, instead of replacing discrete variables with continuous ones as done above, the authors use the so-called "Big-M" method. Similarly, a binary encoding scheme for discrete tap control in LTC is proposed in [35] using the "Big-M" approach. Although useful in addressing bi-linearities, Big-M methods are often unable to perform at scale due to numerical issues resulting from very large M values. If M is chosen too small to overcome numerical issues, then the solution becomes sub-optimal [36]. A linear approximation of an MIP-DOPF problem with LTCs using a piece-wise linear (PWL) approximation is presented in [37]–[41]. This approach increases the number of discrete variable, which significantly limits computational tractability. A variation of MIP-DOPF is developed in [42] using first order approximation of power flow equations to reduce computational complexity.

Contributions: Building independently on the concepts proposed in [43], [44] and adapting them to the conic DOPF models [25]–[28], this paper develops a mixed-integer second order cone program (MISOCP) formulation of DOPF model. We use the relaxed SOCP model and extend it by incorporating the proposed nonlinear LTC model. Non-linear LTC model of the voltage regulator is reformulated as a conic model with second-order conic constraints. The bi-linearity in the proposed model is addressed by using McCormick envelopes [45], which are convex. The effectiveness of the McCormick envelopes have been demonstrated in other optimization models in [43], [44]. To ensure tightness of the proposed convex McCormick relaxations, we also propose an iterative algorithm called Sequential Bound Tightening Algorithm (SBTA) to solve the MISOCP-DOPF model. The algorithm iteratively shrinks the solution space by tightening the convex envelopes. This iteratively improves McCormick approximation. As a result, the proposed model together with the SBTA algorithm outperforms the commercial MINLP solvers in terms of computational time and optimality. Specifically, the major contributions of the paper are:

- The presented MISOCP formulation extends state-of-the-art SOCP models to include discrete control through conic constraints formulation. The bi-linear constraints are then relaxed into a linear framework using McCormick relaxations.
- Development of a novel and efficient SBTA for solving the proposed DOPF problem. The algorithm tightens the relaxation by sequentially reducing the variable bounds. This guarantees an integer-feasible DOPF solution.

In contrast to related work (e.g., [43]), the work herein *(a)* avoids a piece-wise linear approximations that adds up additional complexity due to extra integer variables, *(b)* assesses the feasibility of MISOCP-DOPF solutions to the original MINLP version, and *(c)* ensures a tight relaxation of the MISOCP-DOPF models through SBTA.

MISOCPs can be solved efficiently using extensions of MIP algorithms [46] with computational efficiency similar to MIPs [23] which permits one to leverage the maturity of the existing MIP solvers [47]. In addition, modelling the discrete controls exactly with integer variables overcomes the need for an ad-hoc heuristic rounding schemes.

The rest of the paper is organized as follows. Section II describes the overview of non-convex and convex models of DOPF. Section III describes the proposed MISOCP-DOPF model and the SBTA. Case studies are carried out on 4-node, 69-node, and 137-node distribution feeders in section IV to demonstrate the efficacy of the proposed model and algorithm. Section V provides conclusions and future work.

II. Distribution Optimal Power Flow Formulation Basics

A. Non-Convex Formulation and Rounding Heuristics

1) Non-convex DOPF formulation: The mathematical model of the DOPF can be concisely stated as an objective function, followed by several linear and non-linear constraints. Without loss of generality, for an arbitrary single-phase distribution network, let the nodes in that network be indexed by i and j. y_{ij} is complex admittance, for the branch between node i and node j. Shunt admittance at arbitrary node j is represented by g_j. For every line segment between node i and node j, the real power flowing from i to j be denoted by P_{ij} and reactive power by Q_{ij}, $S_{ij} = P_{ij} + \mathbf{i}\,Q_{ij}$. p_i and q_i denote real and reactive injections at node i. Voltage at each node is given by V_i. $\underline{V_i}$ and $\overline{V_i}$ are the lower and upper voltage limits on the i^{th} node. Similarly, the limits on real power are given by $\underline{P_i}$ and $\overline{P_i}$, and the reactive powers are constrained by the lower limit $\underline{Q_i}$ and upper limit $\overline{Q_i}$. The DOPF can mathematically be stated as,

$$\underset{V,p,q}{\text{argmin}} \quad f(V,p,q) \tag{1a}$$

$$\text{subject to}: \quad S_{ij} = V_i\,(V_i^* - V_j^*)\,y_{ij}^* \tag{1b}$$

$$\sum_{j:i\rightarrow j} P_{ij} = p_i \tag{1c}$$

$$\sum_{j:i\rightarrow j} Q_{ij} = q_i \tag{1d}$$

$$\underline{P_i} \leq p_i \leq \overline{P_i} \tag{1e}$$

$$\underline{Q_i} \leq q_i \leq \overline{Q_i} \tag{1f}$$

$$Q_{ij}^2 + P_{ij}^2 \leq \overline{S_{ij}^2} \tag{1g}$$

$$\underline{V_i} \leq |V_i| \leq \overline{V_i}. \tag{1h}$$

The DOPF model is quadratically constrained optimization problem, which is non-convex in nature. The load model used in the above formulation is of constant power type. However, if a generic Constant Impedance, Current or Power (ZIP) load model is considered, then the load model can be written as quadratic function of nodal voltages V_i. Therefore, a comprehensive DOPF formulation becomes an non-linear programming (NLP) problem. If the discrete control of LTC and cap bank models are included, then the DOPF becomes an MINLP. For detailed MINLP-DOPF formulation, please refer to [11].

2) Rounding Heuristics (Motivating Example): The following concise example illustrates the shortcomings of relaxing the integer variables and using rounding heuristics to recover an integer solution for the DOPF problem.

Consider a simple 7.2 kV modified IEEE 4-node example (phase-a only) feeder as shown in Fig. 1a. The circuit has a voltage regulator ②–③ with a 32-step LTC (tap positions: -16 to 16) and impedance of $(0.47+\mathbf{i}\ 1.05)\,\Omega$, feeder sections ①–② and ③–④ with impedances of $(0.46+\mathbf{i}\ 1.07)$, $(0.47+\mathbf{i}\ 1.07)\,\Omega$, respectively, and a load of $(80 + \mathbf{i}\ 35)$ kVA at node ④. The operating constraints are: 0.95×7.2 kV$\leq |V_4| \leq 1.05 \times 7.2$ kV. The DOPF objective is set to minimize losses. Then, consider the resulting MINLP formulation of the DOPF problem and relax the discrete LTC's integer decision variables with a continuous representation. This results in an NLP-DOPF. We will now investigate what happens under different load conditions when you round up and down a relaxed solution to recover a realizable integer solution.

First, consider rounding up: If the substation voltage is set to $|V_1| = 1.00 \times 7.2$ kV and the load is assumed to be constant PQ. Solving the NLP-DOPF yields an optimal tap position of 8.68. When rounded up to 9.0, this DOPF model becomes infeasible as $|V_4| = 1.053\times 7.2$ kV exceeds its upper bound (please see Fig. 1b). This is because the optimal solution seeks to push the voltage towards its upper bound to reduce the current (and resulting losses) in the constant power load. In this particular case, it is necessary to round down to create a feasible integer solution.

However, just rounding down is not a viable general strategy either. Consider the effects of rounding down on feasibility when the loads is changed to constant-impedance. Now, solving the loss-minimization DOPF yields an optimal tap position of -7.36. When rounded down to the nearest integer -8.0, the DOPF model becomes infeasible as $|V_4| = 0.946 \times 7.2$ kV crosses the lower bound. This is because the LTC keeps the voltage close to the lower bound to reduce the current. Thus, rounding up produces an integer-feasible solution.

By extension of the above, it is clear that for different load types at a single node or multiple nodes, simple rounding heuristics cannot guarantee feasibility. Indeed, we tested the rounding heuristic in a 137-node feeder with 8 LTCs, and we observed similar infeasibility issues. Moreover, the combinatorial nature of the rounding heuristic prohibits scalability in practical-sized feeders.

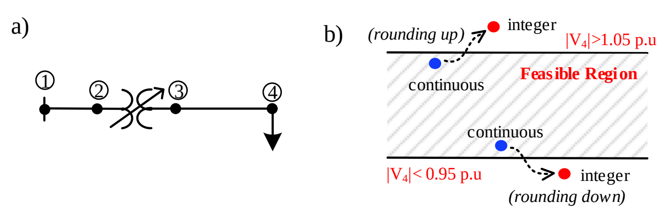

Fig. 1. a) 4-node example feeder to demonstrate the issues of rounding, b) feasible space showing the issues of rounding up and down integer variables.

B. Second-Order Conic Programming Model

The relaxed SOCP-DOPF model in [25]–[28] is based on the branch flow model originally described in [21], [22]. The relaxed DOPF model is developed based on the assumptions that *(i)* the network is connected, *(ii)* the objective function is convex and is strictly increasing on the square of the branch currents, non-increasing with constant power loads, and is independent of apparent power flow on the branches, and *(iii)* the objective function is independent of voltage angles and current angles. These assumptions are quite reasonable for distribution systems. The relaxed DOPF model uses:

(i) Angle Elimination that removes the phase angles of the complex voltages and currents in the original branch flow model. The model is still non-convex due to quadratic equality constraints, which relate real and reactive powers on the branch to respective nodal voltages and branch current. In the *(ii)* Conic Relaxation the quadratic equalities are relaxed with inequalities, which turns the problem into a convex, conic optimization problem. Specifically, the DOPF model can now be modelled as an SOCP if the objective function is linear. In radial distribution systems, the relaxations are proven to be exact under the conditions that there are no upper bound on the loads [27], [28].

Let $G = (N, E)$ be the connected graph of the singe phase distribution system and we assume G is acyclic. N represents the buses and E represents each link in the graph, which are segments of the distribution feeder. The nodes of the network are indexed by i, $N = \{i : i = 1, 2, 3, ..., n\}$. For a distribution system, the root node or the node $i = 1$ is the substation node. Each link in the network is denoted by its two end nodes (i, j), i.e., $i \to j$ denotes a feeder segment from node i to node j. Let E be the set that contains all the links or feeders segments in the network. For each link $(i, j) \in E$, I_{ij} represents current in the branch flowing from node i to node j. Let $l_{ij} = |I_{ij}|^2$ and $v_i = |V_i|^2$. All other notations are as described in the Section II-A. The SOCP-DOPF can be modeled as,

$$\underset{S,l,v,s_1}{\text{argmin}} \quad f(S, l, v, s_1) \tag{2a}$$

$$\text{subject to :}$$

$$p_j = \sum_{k:j\to k} P_{jk} - \sum_{i:i\to j} (P_{ij} - r_{ij}\, l_{ij}) + g_j\, v_j \quad \forall j \tag{2b}$$

$$q_j = \sum_{k:j\to k} Q_{jk} - \sum_{i:i\to j} (Q_{ij} - x_{ij}\, l_{ij}) + b_j\, v_j \quad \forall j \tag{2c}$$

$$v_j = v_i - 2\,(r_{ij}\, P_{ij} + x_{ij}\, Q_{ij}) + (r_{ij}^2 + x_{ij}^2)\, l_{ij} \quad \forall (i,j) \in E \tag{2d}$$

$$\left\| \begin{matrix} 2P_{ij} \\ 2Q_{ij} \\ l_{ij} - v_i \end{matrix} \right\|_2 \le l_{ij} + v_i \qquad \forall (i,j) \in E. \tag{2e}$$

The above formulation is SOCP if the objective function is linear. The conic relaxation is exact if there are no upper bounds on loads. For radial networks, it is possible to recover the angles of voltage and current using an angle recovery algorithm outlined in [27].

III. Proposed Model

In this section, we augment the SOCP-DOPF problem to include a discrete LTC model of the voltage regulator, with the tap position as a control variable. The overall flowchart is shown in Fig. 2. The variables and symbols used in the proposed model are described in the previous Section.

A. Conic Reformulation of LTC Model

For the LTC between nodes i and j, let the primary and secondary voltages be given by v_i and v_j, respectively. Modeling the LTC includes its impedance, which is represented similarly to the impedance of a feeder section.

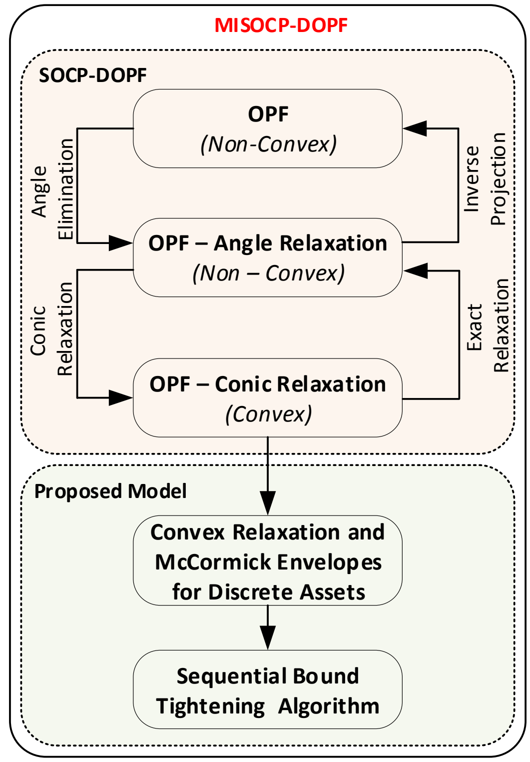

Fig. 2. Overall flowchart of the proposed model integrated to SOCP-DOPF.

Define ΔS as the step change in voltage for a unit change in tap position, t_{ij}. Thus, voltage magnitude control of LTCs can be mathematically modelled as,

$$v_j = v_i\, {k'_i}^2 \tag{3a}$$

$$k'_i = 1 + \Delta S\, t_{ij}. \tag{3b}$$

Defining $k''_i := \frac{1}{v_i}$ and relaxing the above yields:

$$v_j\, k''_i \ge {k'_i}^2 \tag{4a}$$

$$v_i\, k''_i \ge 1. \tag{4b}$$

Inequalities (4a) and (4b) are hyperbolic constraints, which can be reformulated as second-order cone constraints and represent convex sets [48]. However, (4a) and (4b) are lower bounded but have no upper bounds. Relaxations of equality on (3a) and (3b) to inequality in (4a) and (4b) may lead to errors in the model. To ensure tightness of these relaxations and to reduce the modeling errors, we must establish upper bounds on these bi-linear terms and bring the lower and the upper bounds closer. This is achieved by using upper bounds of McCormick envelopes (described next) constructed on the bi-linear terms $v_j k''_i$ and $v_i\, k''_i$. The lower bounds are defined by the second order reformulations of the hyperbolic constraints. Then, we use the SBTA algorithm, to shrink the feasible space using a series of upper bounds.

B. McCormick Envelopes

The bi-linear terms in the proposed model are relaxed by using the McCormick envelopes [45]. This approach allows the formulation to be linearly convex in spite of the nonlinear equations for a regulator's voltage control. In general, the McCormick envelopes for an arbitrary bi-linear term xy with variables $x \in [x^L, x^U]$ and $y \in [y^L, y^U]$ can be mathematically described as,

$$\text{McCormick}\left(x, y, [x^L, x^U], [y^L, y^U]\right) :=$$

$$\text{Upper Bounds} \begin{cases} w \geq x^L y + y^L x - x^L y^L \\ w \geq x^U y + y^U x - x^U y^U \end{cases}$$

$$\text{Lower Bounds} \begin{cases} w \leq x^L y + y^U x - x^L y^U \\ w \leq y^L x + x^U y - y^L x^U. \end{cases} \tag{5}$$

Note that the McCormick envelope represents an inexact envelope approximation of the original NLP problem, which means that the resulting solution may not be realizable without violating the physical constraints. Furthermore, the accuracy of the approximation is directly related to the ranges of the variables (e.g., $x^U - x^L$ and $y^U - y^L$). That is, smaller ranges yield more accurate McCormick envelopes.

C. Mixed-integer Second-order Conic Programming Model of Distribution Optimal Power Flow

The complete modelling of MISOCP-DOPF can be obtained as follows. The notations used in the formulation has same meaning as above. In addition, each of the links that host LTC is contained in the E', $E' \subseteq E$. Without loss of generality, we index all load serving nodes $N_L \subseteq N$. Tap positions for each of the LTCs and voltage for each of the nodes with loads are constrained within an interval.

$$\mathcal{P}: \underset{S,l,v,s_1,t}{\text{argmin}} \quad f(S, l, v, s_1, t) \tag{6a}$$

$$\text{subject to}:$$

$$\text{2(b), 2(c), 2(d), 2(e)}$$

$$\underline{t}_{ij} \leq t_{ij} \leq \bar{t}_{ij} \qquad \forall (i,j) \in E' \tag{6b}$$

$$\underline{v_i} \leq v_i \leq \overline{v_i} \qquad \forall i \in N_L \subseteq N \tag{6c}$$

$$k'_i = (1 + \Delta S\, t_{ij}) \qquad \forall (i,j) \in E' \tag{6d}$$

$$\left\| \begin{matrix} 2k'_i \\ v_j - k''_i \end{matrix} \right\|_2 \leq v_j + k''_i \qquad \forall i \in E' \tag{6e}$$

$$\left\| \begin{matrix} 2 \\ k''_i - v_i \end{matrix} \right\|_2 \leq k''_i + v_i \qquad \forall i \in E' \tag{6f}$$

$$\text{McCormick}\left(k''_i, v_i, [k''^L_i, k''^U_i], [v_i{}^L, v_i{}^U]\right) \qquad \forall i, j \in N \text{ s.t. } (i,j) \in E' \tag{6g}$$

$$k'_i \geq 1 + 2\Delta S t_{ij} + \Delta S^2 t_{ij} \underline{t_{ij}} \qquad \forall (i,j) \in E' \tag{6h}$$

$$t_{ij} \in \mathcal{Z} \qquad \forall (i,j) \in E'. \tag{6i}$$

Thus, the proposed mixed-integer convex model is MISOCP. The optimization objective function focuses on minimizing losses, which yields an objective function that is linear, due to modelling choices.

D. Sequential Bound Tightening Algorithm

Recall that McCormick relaxations, in the proposed LTC model may not be tight. This can be fixed by tightening the convex envelopes for obtaining the desired accuracy in the results. Tightening the McCormick envelopes also reduces computational time. To improve the tightness of the McCormick envelopes in the MISOCP-DOPF model, we propose an iterative Sequential Bound Tightening Algorithm (SBTA) that shrinks the feasible convex region every iteration by decreasing the feasible range of values that each variable can take. This directly reduces the error in the solutions, and after a few iterations, the relaxations are within the feasibility tolerance limit, and the algorithm terminates.

The algorithm is outlined in pseudo code (see Algorithm 1) and iterates while seeking to minimize the approximation error. The tolerance on the error can be set by the user: a tighter tolerance would mean more iterations, while a lower tolerance value could give inaccurate solution.

In the first iteration, the relaxed MISOCP-DOPF model is solved with the initial bounds on variables. In the subsequent iterations, the range of the LTC voltages is set 2ζ around the optimal solution obtained in the previous iteration. Progressively reducing ζ shrinks the solution space every iteration. ϵ is the desired solution quality or the tolerance. In the algorithm, v_i^{*m} and t_{ij}^{*m} denote the optimal values of voltage and optimal integer tap position obtained after solving the problem in the m^{th} iteration.

Algorithm 1 Sequential Bound Tightening Algorithm

Ensure: $\underline{t_n} \leq t_n \leq \overline{t_n} \quad \forall t_n \in \mathcal{M}$
  $\underline{v_i} \leq v_i \leq \overline{v_i} \quad \forall i \in E'$
  initialization $m = 1$
  repeat
    if $m = 1$ **then**
      $\underline{v_i} \leq v_i^m \leq \overline{v_i} \quad \forall i \in E'$
      $1/\overline{v_j} \leq k''^m_i \leq 1/\underline{v_j} \quad \forall i \in E'$
      Solve $\mathcal{P}$
    else
      $v^{*m-1}_i - \zeta^m \leq v_i^m \leq v^{*m-1}_i + \zeta^m \quad \forall i \in E'$
      $1/(v_j^{*m-1} + \zeta^m) \leq k''^m_i \leq 1/(v_j^{*m-1} - \zeta^m) \quad \forall i \in E'$
      Solve $\mathcal{P}$
    end if
    $m = m + 1$
  until $|v_j^* - v_i^*(1 + \Delta S\, t_n^*)^2| \leq \epsilon$

E. Prescribed Tolerance

The approximation error due to McCormick relaxation plays an important role in the convergence of the proposed algorithm. Theoretically, ϵ should be zero or as close to zero as possible. However, this might take a lot of time to converge due to an overwhelmingly large number of iterations. This leads to a trade off between convergence time and accuracy. Our analysis shows that accuracy does not improve considerably when the error drops below a certain value. Since the LTCs are discrete control variables, setting an error tolerance ϵ such that the change in LTC through the SBTA iteration is less than ± 1, the quality of the solution is preserved and the tap position remains unchanged. However, ϵ larger than that can cause change in LTC positions, which causes the solution to drift away from the optimal. Therefore, we conducted a test by changing the input voltage of the LTC in the range $0.9 - 1.1$ p.u, and observing the change in the output voltage of the LTC with the tap setting incremented/decremented by 1. Figure 3 depicts the relationship between change in output voltage due to 1 step change in LTC position, i.e., $\Delta V = |(v_j - v_i(1+\Delta S\, t_n)^2) - (v_j - v_i(1 + \Delta S\,(t_n \pm 1))^2)|$ and the input voltage to the LTCs. We choose ϵ be less than ΔV. We found that setting $\epsilon \leq 0.0102$ gives sufficiently accurate results.

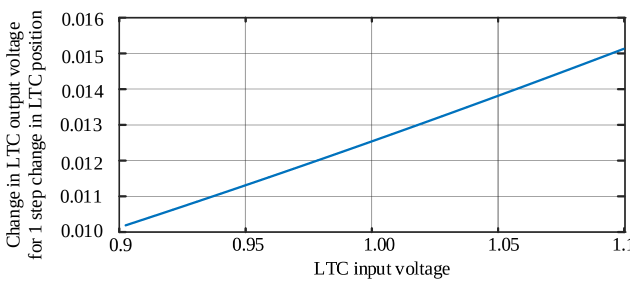

Fig. 3. Upper bound on ϵ at different primary voltages.

IV. Numerical Simulations

In this section, performance of the proposed DOPF model is analyzed through a series of numerical simulations. Starting with a modified IEEE 4-node system, we run a loss minimization problem with LTC tap positions as decision variables. Results of this study are extensively analyzed and the feasibility of the obtained optimal solutions are compared with the solutions of MINLP-DOPF. Case studies using an 4-node feeder with one LTC, 69-node feeder with 4 LTCs are carried out to illustrate the relative scalability of the MISOCP model compared to the MINLP-DOPF. In addition, to reflect European operational practice, case studies on the 69-node test feeder have been included to consider voltage bounds of $\pm 10\%$ around nominal, which further reduced losses. We also explore alternative objective functions and include continuous control resources as decision variables in the MISOCP formulation. Finally, we perform a multi-period DOPF on 137-node feeder with 8 LTCs to highlight the scalability of the proposed model in comparison to contemporary MINLP modelling. Note that LTCs in all these studies have 32 steps, with step size ΔS=0.00625.

A. Loss Minimization

1) 4-node Feeder: We take phase-a of the IEEE 4-node three-phase distribution feeder, as shown in Fig.1a, to demonstrate working of MISOCP-DOPF and SBTA. As shown in the Fig.1a, we added one voltage regulator with LTC. We minimize the power losses on the modified 4-node feeder with the tap position of the LTC as the discrete control variable. We vary load connected at node-4 to demonstrate tightness of the solutions for various loading condition. The load power factor is fixed at $cos(tan^{-1}(1/3))$.

Since only node 4 has connected loads, the voltage on this node is constrained between 0.95 and 1.05 pu. Voltages at all other nodes are unconstrained and free to take any values. Tap positions of the LTC is constrained between -16 and $+16$ as discrete integers. MOSEK is used to solve the MISOCP-DOPF model. Voltage at node 1 is fixed at 1.0 pu. The algorithm terminates when the ϵ is less than 0.0102.

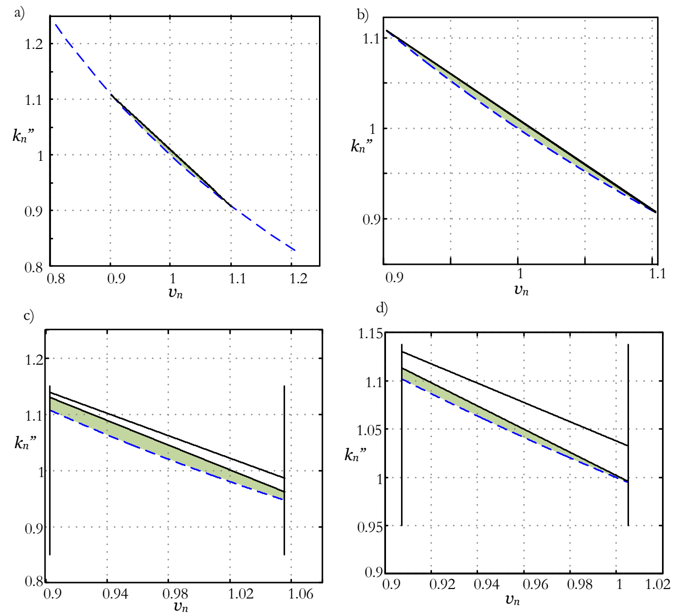

Fig. 4. Convex relaxation of nonlinear constraint $v_n k_n^{"} = 1$ over the region $v_n \in [0.95, 1.05]$ and $k_n^{"} \in [\frac{1}{1.05}, \frac{1}{0.95}]$. The set of points satisfying the constraints are represented by a dotted line. The boundaries of the McCormick envelopes are shown in bold, which encloses the (shaded) convex region.

The proposed SBTA for all load cases converged in 2 iterations. The voltage bound in Algorithm 1 at the first iteration is set at 0.1, and $\zeta^{[2]} = 0.03$ is used in the second iteration (see Fig. 5). Fig. 5 also shows reduction on voltage error over the iterations of the SBTA algorithm. Decreasing ζ each iteration reduces the convex feasible set (with bounds from the McCormick envelopes). The shrinkage can be seen in 5, the right axis shows the difference between the upper bound and the lower bound of the input voltage to the LTC. This reduction in the feasible set is essential to achieving tightness on the solutions. Fig. 4 shows the effect of decreasing ζ on the convex feasible region. Fig. 4a shows the overall

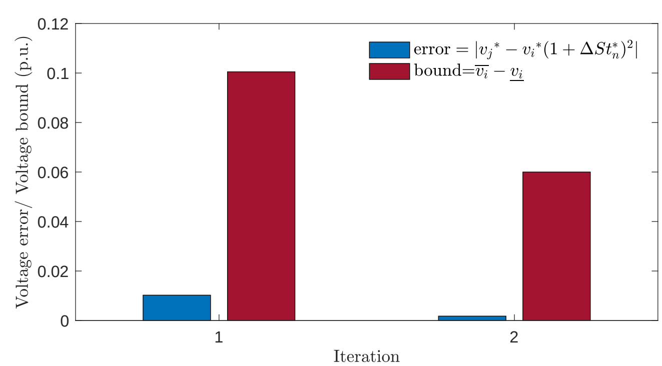

Fig. 5. Error on voltage calculation and voltage bound decrements over the iterations in SBTA.

function together with the McCormick envelopes. Fig. 4b is the zoomed in version around McCormick envelopes. In Fig. 4c and 4d, the bounded area is shrinking. The convergence of the model depends on the value of ζ and the tolerance of error ϵ. For a smaller ζ and a smaller tolerance ϵ, the model would require more iterations to achieve convergence. However, ζ should be selected such that 2ζ is less than the difference of upper and the lower bounds on the variables in the McCormick envelopes.

Table I summarizes the results under different loading conditions. As the load increases, the losses increase quadratically. The LTC position increases with an increase in the load to minimize the current in order to reduce the losses and keep the voltage at node ④ within acceptable limits. All of the solutions in Table I are feasible with respect to the original MINLP-DCOPF model.

TABLE I
RESULTS FOR THE MODIFIED 4-NODE FEEDER WITH AN LTC.

Load		Voltage, pu			Loss	LTC
kW	kVar	Node②	Node③	Node④	kW	
300	100	0.99904	1.05075	1.04999	2.639	9
450	150	0.99650	1.05302	1.04999	5.999	9
600	200	0.99390	1.0552	1.04999	10.775	11
750	250	0.99121	1.05760	1.04999	17.015	12
900	300	0.98845	1.05991	1.04999	24.768	14
1050	350	0.98560	1.06221	1.04999	34.092	15
1200	400	0.98266	1.06333	1.04878	45.077	16
1500	500	0.97646	1.04980	1.03023	72.983	16
1800	600	Infeasible				

2) 69-node Feeder: A loss minimization study on a single phase 69-node system was also carried out using discrete LTC positions as control variables. Schematic of 69-node system is shown in Fig. 6 [49]. The feeder is modified by adding 4 LTCs. The intention of this study is to demonstrate scalability of the proposed model. The total losses in the system is found to be 192.54 kW. The LTC positions are 8,6,0, and 9, respectively. The voltage profile along the feeder is shown in Fig 7. The feasibility of the obtained optimal solution was confirmed with basic power flow analysis. Solving MINLP-DOPF model yields total losses of 206.50 kW.

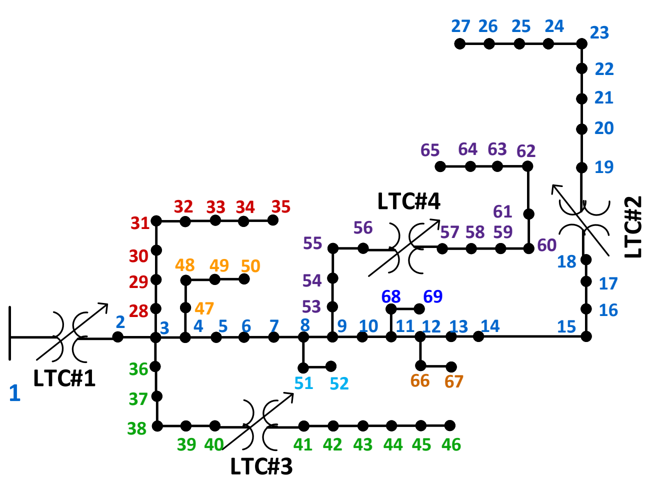

Fig. 6. Schematic of modified 69-node feeder with 4 LTCs.

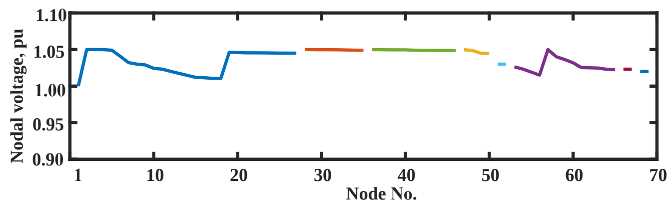

Fig. 7. Nodal voltage profile on 69-node feeder.

In order to further test the effectiveness of the proposed model to networks with a higher permissible voltage deviation, we allowed the voltage limit in the range 0.9 to 1.1 p.u. We run this simulation with the same loss minimization objective. As we expected, the total active loss drops from 192.8 kW to 173.89 kW. The obtained solutions are tight with respect to the nonlinear DOPF in (1b) and (1h).

3) 137-node Feeder: A 137-node distribution system is used to solve the MISOCP-DOPF to demonstrate the scalability of the method. The network is synthesized by cascading two sections of a 69-node feeder in Fig. 6 (second feeder connected at node 69 of the first feeder) and load scaled by a factor of 0.5. The resulting feeder has total of 8 LTCs. We minimize the active power losses on the network by controlling the tap positions of the LTCs. Nodal voltages are constrained between 0.95 and 1.05 p.u and LTCs between -16 and $+16$. The results are summarized in Table II. All the voltages match closely (up to three decimal places) with the solutions obtained from power flow analysis. We only present the voltages on the nodes which are connected to the LTCs. The total losses on the system are 279.2 kW and the proposed MISOCP-DOPF model took 16.38 seconds to solve using MOSEK. Using the commercial KNITRO solver for the MINLP-DOPF formulation took 27.6 seconds with total losses of 298 kW. We run the MINLP-DOPF model with MISOCP-DOPF solution as initial condition to ensure that MINLP-DOPF is not trapped at local solutions. This approach did speed up the computation time of MINLP-DOPF but the objective function value did not change in MINLP-DOPF. The difference in the objective function values in MISOCP-DOPF and MINLP-DOPF are due to the difference in convergence set in the solvers and due to the relaxations in SOCP model. We run several cases by varying the load, and we observed that MISOCP-DOPF solutions are always feasible to MINLP-DOPF, and MISOCP-DOPF solutions are superior to those of MINLP-DOPF.

TABLE II
SUMMARY OF THE RESULTS FOR THE 137-NODE DISTRIBUTION FEEDER.

LTC	Connecting Nodes		Tap	Voltages, p.u	
	Sending	Receiving		Sending	Receiving
LTC#1	1	2	8	1.00000	1.04993
LTC#2	18	19	9	0.98892	1.04743
LTC#3	40	41	-2	1.04968	1.04036
LTC#4	56	57	6	1.01736	1.04901
LTC#5	69	70	11	0.98248	1.04999
LTC#6	86	87	3	1.03059	1.04969
LTC#7	108	109	-1	1.04979	1.04490
LTC#8	124	125	4	1.03315	1.04999

TABLE III
DISPATCH OF PV AT VARIOUS NODES ACROSS THE 69-NODE TEST FEEDER.

Node	PV Capacity (KVA)	Dispatch	
		PV (kW)	PV (kVAr)
4	500	372.91	331.94
7	500	405.51	292.50
9	500	406.66	290.89
12	500	396.46	304.65
22	500	380.35	267.53
37	300	141.70	57.46
47	500	392.91	308.61
52	500	404.79	293.49
57	300	118.93	180.44
60	400	327.48	229.67

B. Other Objective Functions

The non-integer part of the developed work is based on recent advancements in SOCP relaxation of OPF problem [27], [28]. An arbitrary choice of objective function may not lead to a tight SOCP solutions; hence, the user must be careful in selecting the objective function in SOCP relaxation of OPF problem. We have tested the model with voltage deviation (i.e, $\sum_i (v_i - v_i^{\text{nom}})^2$) minimization in the objective function, and observed that the solution is not tight [50]. However, with a multi-objective minimization of losses and voltage deviations, we were able to achieve tight and efficient solutions. In fact, with the 69-node network augmented with several solar PV sites (i.e., curtailable generation), simulations demonstrate the combined optimization of continuous and discrete assets within the proposed MISOCP framework. Specifically, the multi-objective simulation converged in four SBTA iterations. The capacity of the solar PV site at each node and corresponding dispatch (active and reactive power) are shown in Table III. The obtained LTC positions are $\{0, 0, -1, 8\}$ and the total active power loss is 90.73 kW.

C. Multi-Period DOPF

We use the proposed model on the 137-node feeder to optimize the tap settings on LTCs over a 24-hour horizon with hourly resolution. The performance of the proposed MISOCP-DOPF model is compared with the results of MINLP model obtained using KNITRO (see Fig. 8).

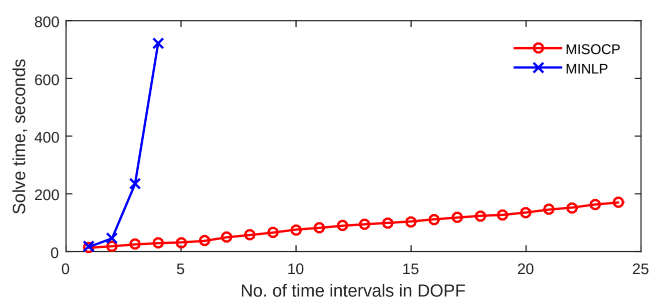

Fig. 8. Solve time from MISOCP-DOPF model vs MINLP model for the multi-period DOPF on 137-node feeder.

The proposed MISOCP-DOPF has better efficiency and scalability. The proposed MISOCP-DOPF model enabled us to solve the entire 24-hour DOPF problem with discrete tap controls in less than 5 minutes. While KNITRO was only able to solve the multi-period MINLP-DOPF model for ≤ 3 time intervals. Other state-of-the-art solvers (e.g. SCIP, BARON, CONUENE) also failed to obtain a solution, when the number of time intervals was increased to ≥ 4. MISOCP-DOPF shows at least one order magnitude faster solve time compared to the MINLP-DOPF formulation. For the multi-period DOPF, using the solution from MISOCP-DOPF as initial condition to the MINLP-DOPF model did not help speed up the computation for ≥ 2 of time intervals. Note that the computational complexity in this work is discussed based on North American type feeders, where multiple in-line LTCs can exist for voltage regulation [51]. In European type networks, since in-line LTCs are not common and LTCs are primarily equipped at substation transformers [52], rounding heuristics could still be a viable option due to small search space.

To compare the quality of the MISOCP solutions, we have evaluated the optimally gap relative to the (integer relaxed) SOCP and observed that the worst-case optimality gap is less than 0.68%, which ensures quality of the MISOCP solution. The MINLP-DOPF showed large optimality gap of less than 6.2%, which is almost ten times larger.

V. CONCLUSION AND FUTURE WORK

In this work, we first demonstrated challenges surrounding infeasibility of the rounding approach in solving DOPF model with LTC control. Then, based on a state-of-the-art SOCP formulation of the DOPF and convex relaxation of the LTC models with McCormick envelopes, we built a MISOCP version of the DOPF model. Then, we applied a sequential bound tightening algorithm to ensure that the relaxations in MISOCP-DOPF are tight. Numerical simulations are presented for 3 distribution feeders and showed that the proposed method is superior to MINLP in all regards. The MISOCP-DOPF model outperforms the MINLP model in terms of solve time and optimality. Also, we demonstrated that the solution from MISOCP model is AC feasible to which ensures that the infeasibility issue due to rounding approach is also overcame through the MISOCP-DOPF formulation.

Future work will investigate analytical conditions that guarantee feasibility and optimality of the proposed MISOCP-DOPF model and comparisons with alternative formulations/solution methods. In addition, we seek to extend this work along the following directions:

- include battery storage and 4-quadrant inverter control into the optimization, which increases temporal coupling due to energy dynamics,
- extend MISOCP to consider the full three-phase OPF models to actively manage phase imbalances.

In this paper, we have successfully demonstrated efficiency and usefulness of MISOCP model in solving optimal power flow problem on distribution feeders with discrete and continuously controllable assets. The proposed model can certainly serve as a core model in ADMS tools.

REFERENCES

[1] G. Schellstede and A. Huber, "A modern distribution management system," in *Proc. 12th International Conference on Electricity Distribution*, May 1993, pp. 4.6/1–4.6/6.

[2] "Distributed System Implementation Plan," Consolidated Edison of New York, Tech. Rep., 2016.

[3] D. J. Rogers, T. C. Green, and R. W. Silversides, "A low-wear onload tap changer diverter switch for frequent voltage control on distribution networks," *IEEE Transactions on Power Delivery*, vol. 29, no. 2, pp. 860–869, Apr. 2014.

[4] K. Turitsyn, P. Šulc, S. Backhaus, and M. Chertkov, "Distributed control of reactive power flow in a radial distribution circuit with high photovoltaic penetration," in *Proc. IEEE PES General Meeting*, July 2010, pp. 1–6.

[5] Y. P. Agalgaonkar, B. C. Pal, and R. A. Jabr, "Distribution voltage control considering the impact of pv generation on tap changers and autonomous regulators," *IEEE Transactions on Power Systems*, vol. 29, no. 1, pp. 182–192, Jan 2014.

[6] J. A. Kersulis and I. A. Hiskens, "Renewable voltage regulation and the transformer tapping trade-off," in *Proc. IEEE Innovative Smart Grid Technologies - Asia*, Nov 2016, pp. 960–965.

[7] J. Lee, *A First Course in Combinatorial Optimization*, ser. Cambridge Texts in Applied Mathematics. Cambridge University Press, 2004.

[8] F. Capitanescu, I. Bilibin, and E. R. Ramos, "A comprehensive centralized approach for voltage constraints management in active distribution grid," *IEEE Transactions on Power Systems*, vol. 29, no. 2, pp. 933–942, March 2014.

[9] W. F. Tinney, J. M. Bright, K. D. Demaree, and B. A. Hughes, "Some deficiencies in optimal power flow," *IEEE Transactions on Power Systems*, vol. 3, no. 2, pp. 676–683, May 1988.

[10] A. D. Papalexopoulos, C. F. Imparato, and F. F. Wu, "Large-scale optimal power flow: effects of initialization, decoupling and discretization," *IEEE Transactions on Power Systems*, vol. 4, no. 2, pp. 748–759, May 1989.

[11] S. Paudyal, C. A. Cañizares, and K. Bhattacharya, "Optimal operation of distribution feeders in smart grids," *IEEE Transactions on Industrial Electronics*, vol. 58, no. 10, pp. 4495–4503, Oct. 2011.

[12] F. Capitanescu and L. Wehenkel, "Sensitivity-based approaches for handling discrete variables in optimal power flow computations," *IEEE Transactions on Power Systems*, vol. 25, no. 4, pp. 1780–1789, Nov. 2010.

[13] S. S. Lin, S. C. Horng, and C. H. Lin, "Distributed quadratic programming problems of power systems with continuous and discrete variables," *IEEE Transactions on Power Systems*, vol. 28, no. 1, pp. 472–481, Feb. 2013.

[14] L. Chen, H. Suzuki, and K. Katou, "Mean field theory for optimal power flow," *IEEE Transactions on Power Systems*, vol. 12, no. 4, pp. 1481–1486, Nov. 1997.

[15] E. M. Soler, E. N. Asada, and G. R. M. da Costa, "Penalty-based nonlinear solver for optimal reactive power dispatch with discrete controls," *IEEE Transactions on Power Systems*, vol. 28, no. 3, pp. 2174–2182, Aug. 2013.

[16] A. Mohapatra, P. R. Bijwe, and B. K. Panigrahi, "An efficient hybrid approach for volt/var control in distribution systems," *IEEE Transactions on Power Delivery*, vol. 29, no. 4, pp. 1780–1788, Aug. 2014.

[17] S. Paudyal, C. A. Cañizares, and K. Bhattacharya, "Three-phase distribution OPF in smart grids: Optimality versus computational burden," in *Proc. Innovative Smart Grid Technologies (ISGT)*, Dec 2011, pp. 1–7.

[18] C. Coffrin and P. V. Hentenryck, "A linear-programming approximation of ac power flows," *INFORMS Journal on Computing*, vol. 26, no. 4, pp. 718–734, 2014.

[19] M. R. Almassalkhi and I. A. Hiskens, "Model-predictive cascade mitigation in electric power systems with storage and renewables—Part I: Theory and implementation," *IEEE Transactions on Power Systems*, vol. 30, no. 1, pp. 67–77, 2015.

[20] R. Palma-Benhke, A. Philpott, A. Jofré, and M. Cortés-Carmona, "Modelling network constrained economic dispatch problems," *Optimization and Engineering*, vol. 14, no. 3, pp. 417–430, Sep 2013.

[21] M. E. Baran and F. F. Wu, "Optimal capacitor placement on radial distribution systems," *IEEE Transactions on Power Delivery*, vol. 4, no. 1, pp. 725–734, Jan. 1989.

[22] M. Baran and F. F. Wu, "Optimal sizing of capacitors placed on a radial distribution system," *IEEE Transactions on Power Delivery*, vol. 4, no. 1, pp. 735–743, Jan. 1989.

[23] J. A. Taylor, *Convex optimization of power systems*. Cambridge University Press, 2015.

[24] D. K. Molzahn and I. A. Hiskens, "Convex relaxations of optimal power flow problems: An illustrative example," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 63, no. 5, pp. 650–660, 2016.

[25] S. H. Low, "Convex Relaxation of Optimal Power Flow—Part I: Formulations and Equivalence," *IEEE Transactions on Control of Network Systems*, vol. 1, no. 1, pp. 15–27, Mar. 2014.

[26] S. H. Low, "Convex relaxation of optimal power flow—Part II: Exactness," *IEEE Transactions on Control of Network Systems*, vol. 1, no. 2, pp. 177–189, Jun. 2014.

[27] M. Farivar and S. H. Low, "Branch flow model: Relaxations and convexification - Part I," *IEEE Transactions on Power Systems*, vol. 28, no. 3, pp. 2554–2564, Aug. 2013.

[28] ——, "Branch flow model: Relaxations and convexification - Part II," *IEEE Transactions on Power Systems*, vol. 28, no. 3, pp. 2565–2572, Aug. 2013.

[29] E. Dall'Anese, H. Zhu, and G. B. Giannakis, "Distributed optimal power flow for smart microgrids," *IEEE Transactions on Smart Grid*, vol. 4, no. 3, pp. 1464–1475, Sep. 2013.

[30] R. A. Jabr, "Radial distribution load flow using conic programming," *IEEE Transactions on Power Systems*, vol. 21, no. 3, pp. 1458–1459, Aug 2006.

[31] L. Gan, N. Li, U. Topcu, and S. H. Low, "Exact convex relaxation of optimal power flow in radial networks," *IEEE Transactions on Automatic Control*, vol. 60, no. 1, pp. 72–87, Jan. 2015.

[32] B. A. Robbins and A. D. Domínguez-García, "Optimal reactive power dispatch for voltage regulation in unbalanced distribution systems," *IEEE Transactions on Power Systems*, vol. 31, no. 4, pp. 2903–2913, Jul. 2016.

[33] B. A. Robbins, H. Zhu, and A. D. Domínguez-García, "Optimal tap setting of voltage regulation transformers in unbalanced distribution systems," *IEEE Transactions on Power Systems*, vol. 31, no. 1, pp. 256–267, Jan. 2016.

[34] T. Ding, S. Liu, W. Yuan, Z. Bie, and B. Zeng, "A two-stage robust reactive power optimization considering uncertain wind power integration in active distribution networks," *IEEE Transactions on Sustainable Energy*, vol. 7, no. 1, pp. 301–311, Jan. 2016.

[35] W. Wu, Z. Tian, and B. Zhang, "An exact linearization method for OLTC of transformer in branch flow model," *IEEE Transactions on Power Systems*, vol. 32, no. 3, pp. 2475–2476, May 2017.

[36] S. Bradley, A. Hax, and T. Magnanti, *Applied Mathematical Programming*. Addison-Wesley Publishing Company, 1977.

[37] B. Liu, F. Liu, S. Mei, and Y. Chen, "Optimal reactive power flow with exact linearized transformer model in distribution power networks," in *Proc. 27th Chinese Control and Decision Conference*, May 2015, pp. 5562–5567.

[38] T. Ding, S. Liu, W. Yuan, Z. Bie, and B. Zeng, "A two-stage robust reactive power optimization considering uncertain wind power integration in active distribution networks," *IEEE Transactions on Sustainable Energy*, vol. 7, no. 1, pp. 301–311, Jan. 2016.

[39] Z. Tian, W. Wu, B. Zhang, and A. Bose, "Mixed-integer second-order cone programing model for var optimisation and network reconfiguration in active distribution networks," *IET Generation, Transmission Distribution*, vol. 10, no. 8, pp. 1938–1946, 2016.

[40] L. H. Macedo, J. F. Franco, M. J. Rider, and R. Romero, "Optimal operation of distribution networks considering energy storage devices," *IEEE Transactions on Smart Grid*, vol. 6, no. 6, pp. 2825–2836, Nov. 2015.

[41] A. Borghetti, "Using mixed integer programming for the volt/var optimization in distribution feeders," *Electric Power Systems Research*, vol. 98, pp. 39 – 50, 2013.

[42] C. Sabillon-Antunez, O. D. Melgar-Dominguez, J. F. Franco, M. Lavorato, and M. J. Rider, "Volt-var control and energy storage device operation to improve the electric vehicle charging coordination in unbalanced distribution networks," *IEEE Transactions on Sustainable Energy*, vol. 8, no. 4, pp. 1560–1570, Oct 2017.

[43] E. Briglia, S. Alaggia, and F. Paganini, "Distribution network management based on optimal power flow: Integration of discrete decision variables," in *Proc. 51st Annual Conference on Information Sciences and Systems (CISS)*, Mar. 2017, pp. 1–6.

[44] F. Wu, H. Nagarajan, A. Zlotnik, R. Sioshansi, and A. M. Rudkevich, "Adaptive convex relaxations for gas pipeline network optimization," in *Proc. American Control Conference (ACC)*, May 2017, pp. 4710–4716.

[45] G. P. McCormick, "Computability of global solutions to factorable nonconvex programs: Part I — convex underestimating problems," *Mathematical Programming*, vol. 10, no. 1, pp. 147–175, 1976.

[46] H. Y. Benson and U. Sağlam, *Mixed-Integer Second-Order Cone Programming: A Survey*, ch. Chapter 2, pp. 13–36.

[47] A. Ben-Tal and A. Nemirovski, "On polyhedral approximations of the second-order cone," *Mathematics of Operations Research*, vol. 26, no. 2, pp. 193–205, 2001.

[48] M. S. Lobo, L. Vandenberghe, S. Boyd, and H. Lebret, "Applications of second-order cone programming," *Linear Algebra and its Applications*, vol. 284, no. 1, pp. 193 – 228, 1998, international Linear Algebra Society (ILAS) Symposium on Fast Algorithms for Control, Signals and Image Processing.

[49] J. S. Savier and D. Das, "Impact of network reconfiguration on loss allocation of radial distribution systems," *IEEE Transactions on Power Delivery*, vol. 22, no. 4, pp. 2473–2480, Oct 2007.

[50] Q. Li and V. Vittal, "Non-iterative enhanced SDP relaxations for optimal scheduling of distributed energy storage in distribution systems," *IEEE Transactions on Power Systems*, vol. 32, no. 3, pp. 1721–1732, May 2017.

[51] K. P. Schneider, B. A. Mather, B. C. Pal, C. W. Ten, G. J. Shirek, H. Zhu, J. C. Fuller, J. L. R. Pereira, L. F. Ochoa, L. R. de Araujo, R. C. Dugan, S. Matthias, S. Paudyal, T. E. McDermott, and W. Kersting, "Analytic considerations and design basis for the IEEE distribution test feeders," *IEEE Transactions on Power Systems*, vol. 33, no. 3, pp. 3181–3188, May 2018.

[52] C. Long and L. F. Ochoa, "Voltage control of PV-rich LV networks: OLTC-fitted transformer and capacitor banks," *IEEE Transactions on Power Systems*, vol. 31, no. 5, pp. 4016–4025, Sep. 2016.

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